# *The Blind Watchmaker Network:*
## *Scale-freeness and evolution*

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# Outline

- Statistical mechanics
- Balls in boxes model
- Variational calculus
- Variational calculus using a random process
- Constrained Balls in Boxes model
- Network constraints
- Comparison with real networks
- Other network properties (C-r space)
- Conclusions

# Statistical mechanics

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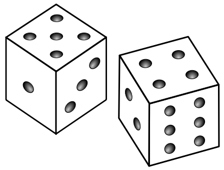

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Distr. 2: 4 states { **AAAB**, **AABA**, **ABAA**, **BAAA**

Distr. 3: 6 states { **AABB**, **ABAB**, **ABBA**, **BABA**, **BAAB**, **BBAA**

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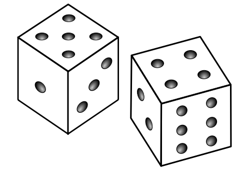

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**Example:** Flip a coin with sides **A** and **B**. How many **A**:s and **B**:s will you most likely have after four flips?

**Answer:** The distribution (number of **A**:s and **B**:s) that has the most number of states (highest entropy) will win (in the long run)!

In this case distribution 3 is most likely to show up (prob = 6/(1+4+6+4+1) = 3/8).

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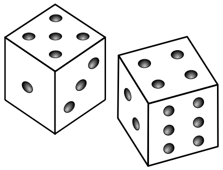
# Network → Balls in boxes model

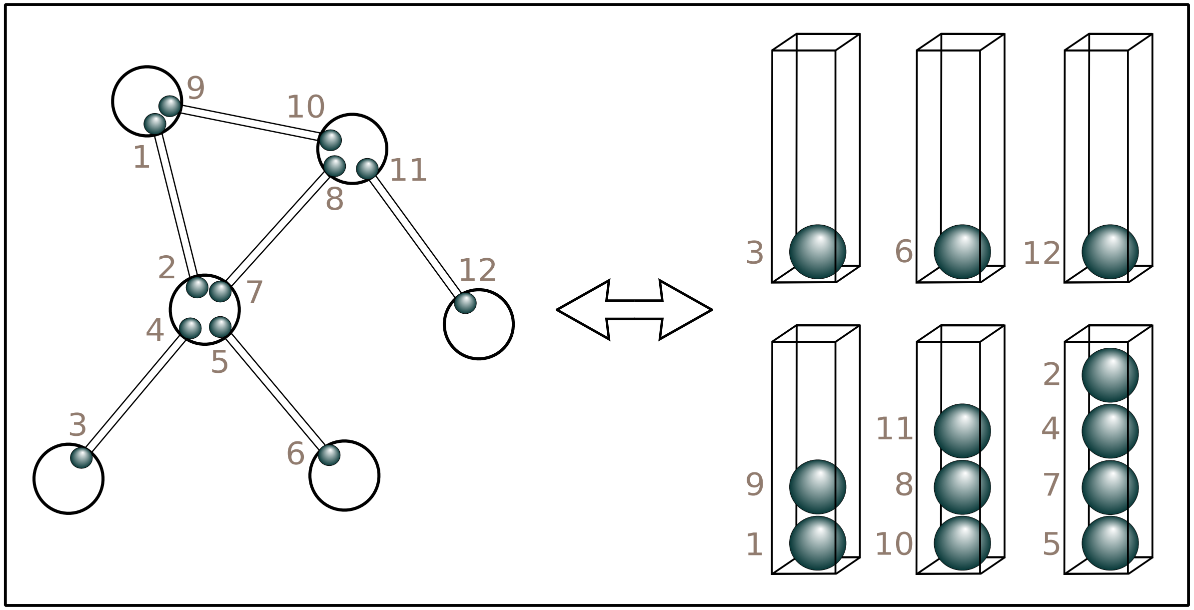


Map a network onto a set of balls and boxes.

Boxes $\Longleftrightarrow$ Nodes

Balls $\Longleftrightarrow$ Link ends

A link is defined by two link ends, e.g. (7,8)



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$$\Rightarrow N(k) = Ae^{-bk}/k! = A\langle k \rangle^k / k! \Rightarrow \text{Poisson distribution (Erdős-Renyi)}$$

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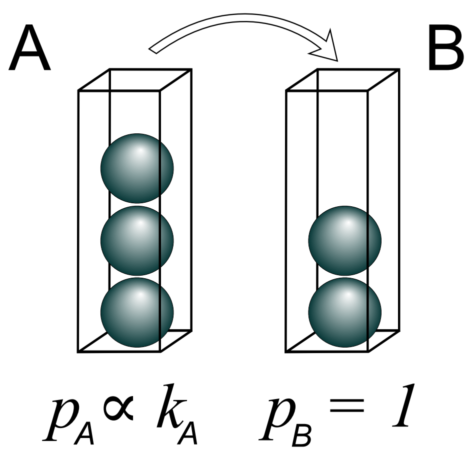

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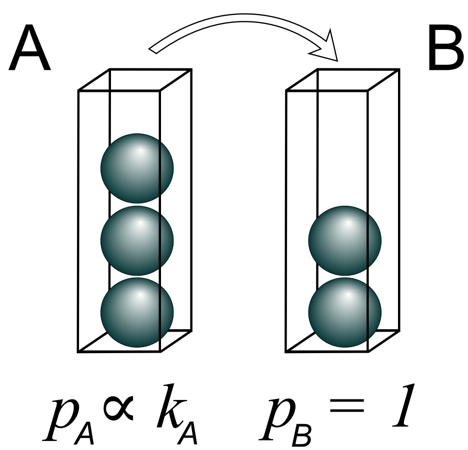


$\Longrightarrow$

**The random process**

1) Pick one box with probability $p \propto k$ and one box with $p = 1/N$.

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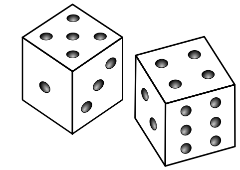

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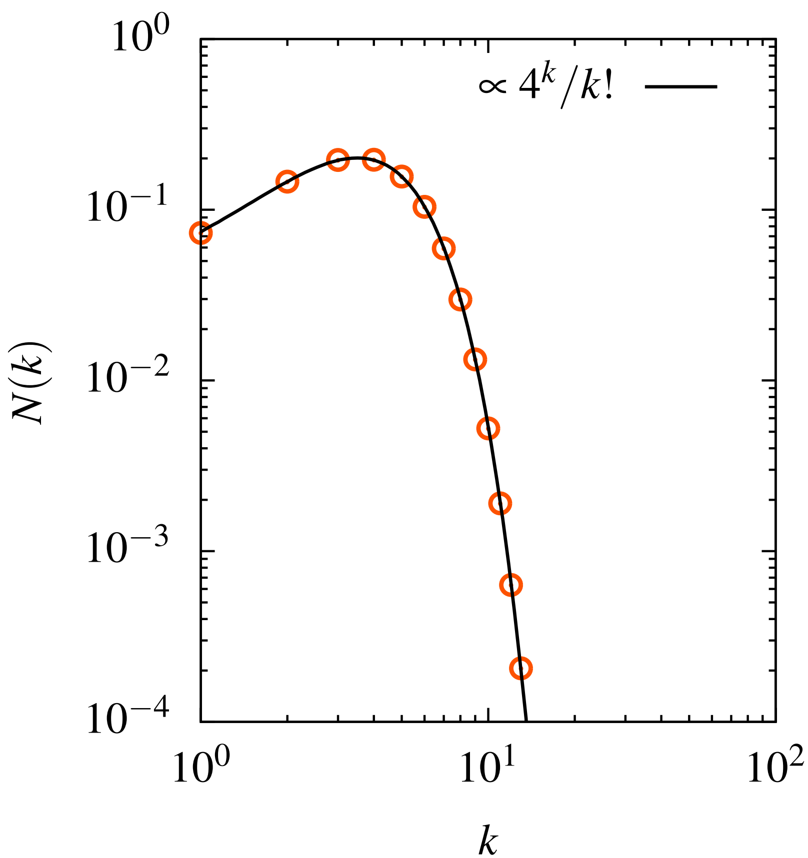


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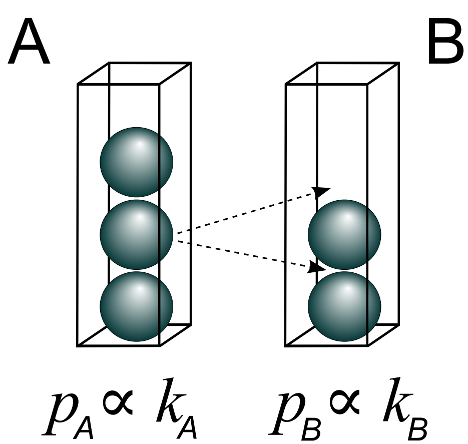

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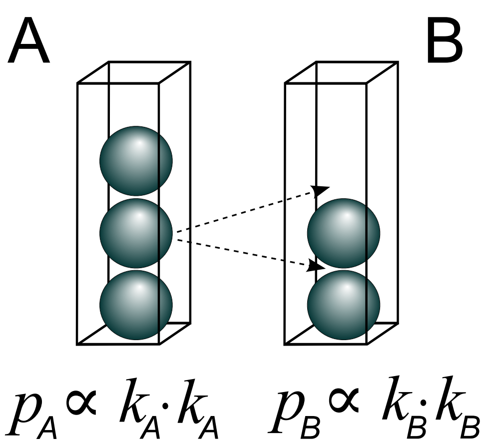

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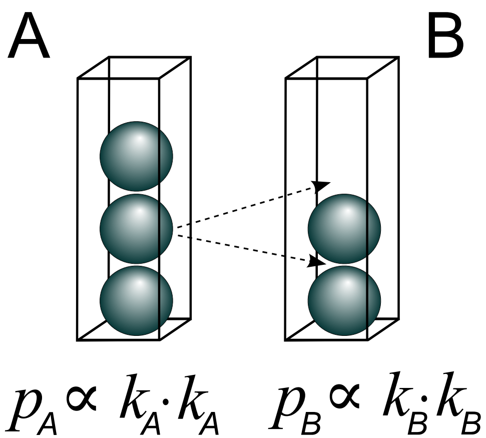


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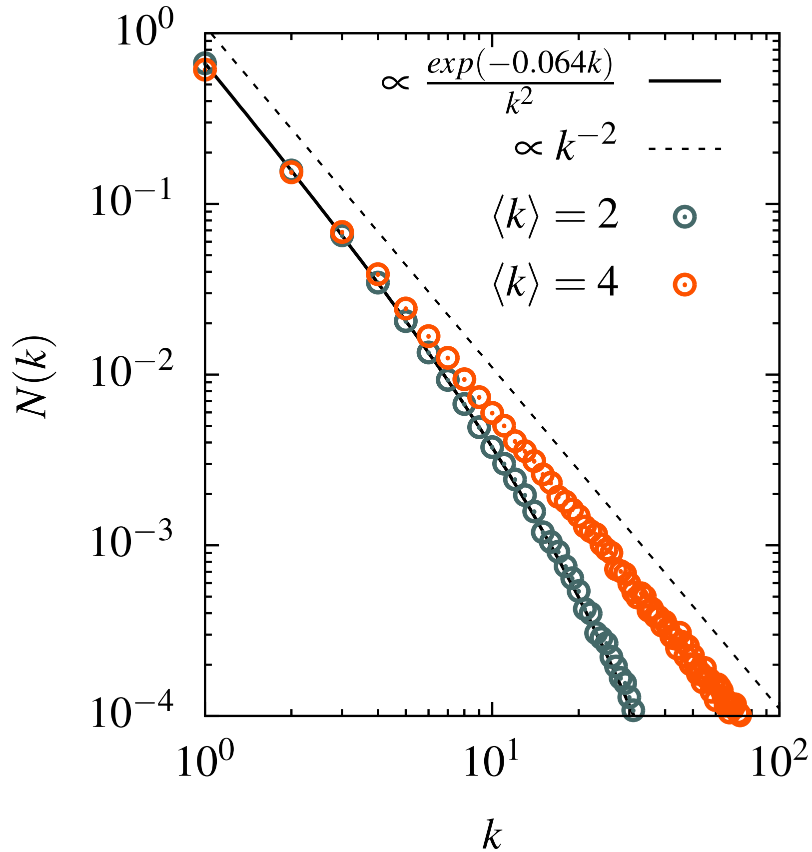


$$\Rightarrow N(k) = Ae^{-bk}/k^2$$

You are not allowed to empty a box!

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## Network constraints

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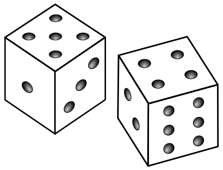

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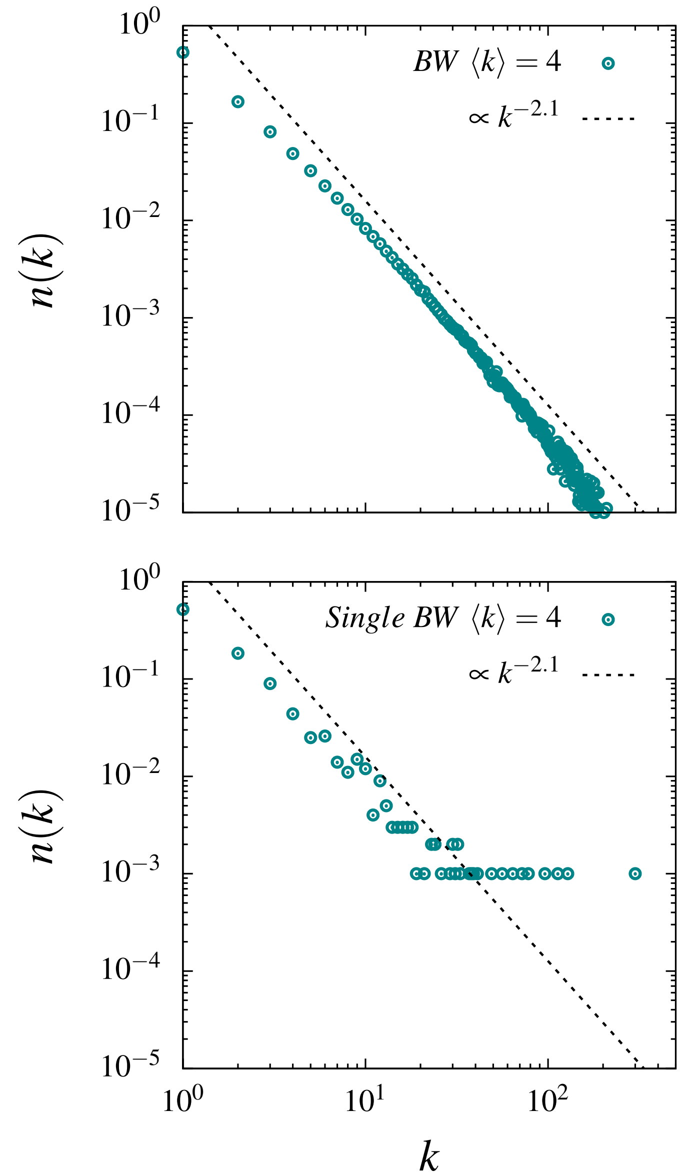

# Comparison with real data

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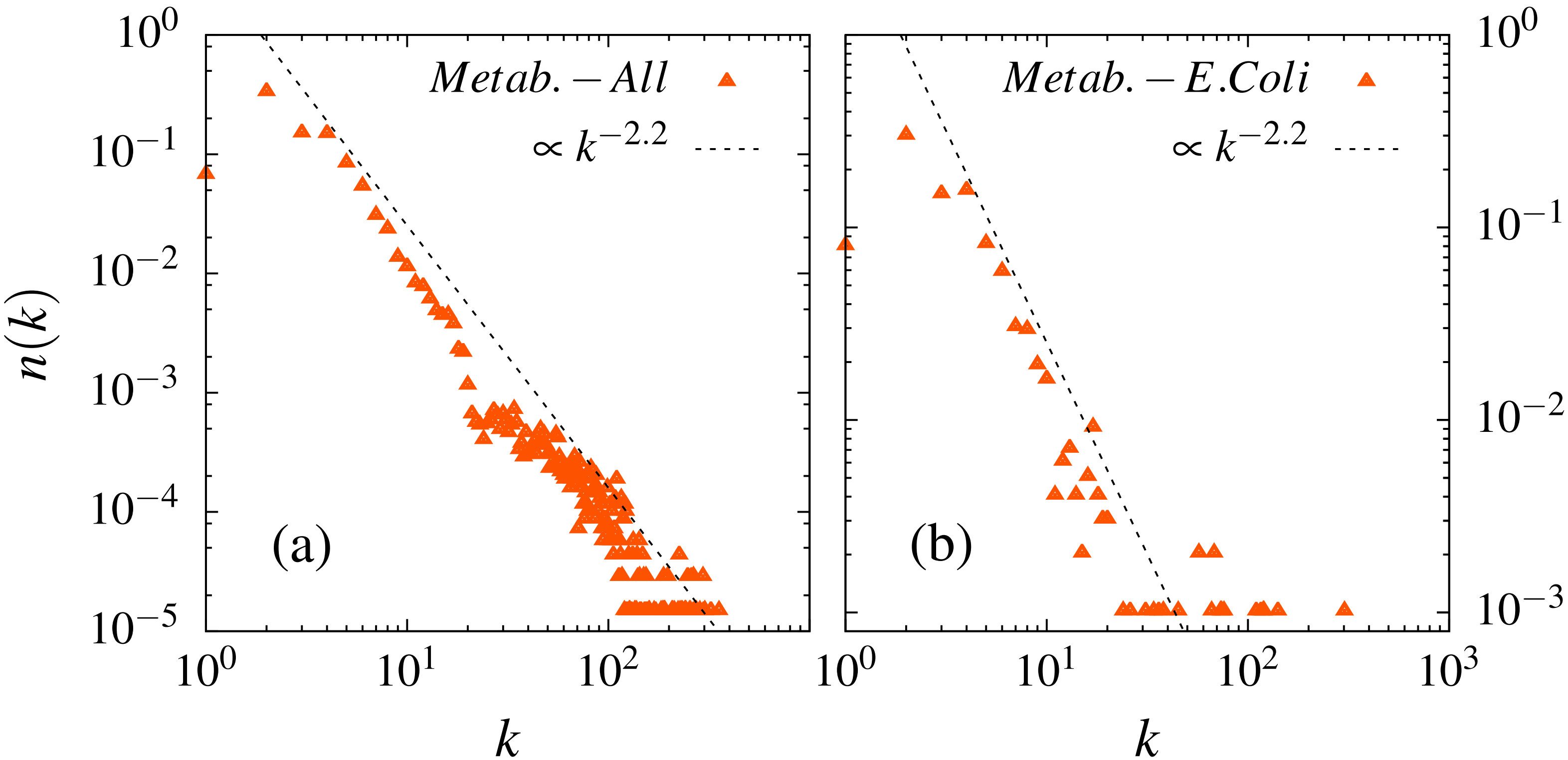
Metabolic networks:

(a) Average over 107 organisms

(b) E. Coli




Ma H and Zeng A-P, Bioinformatics 19: 270-277 (2003).

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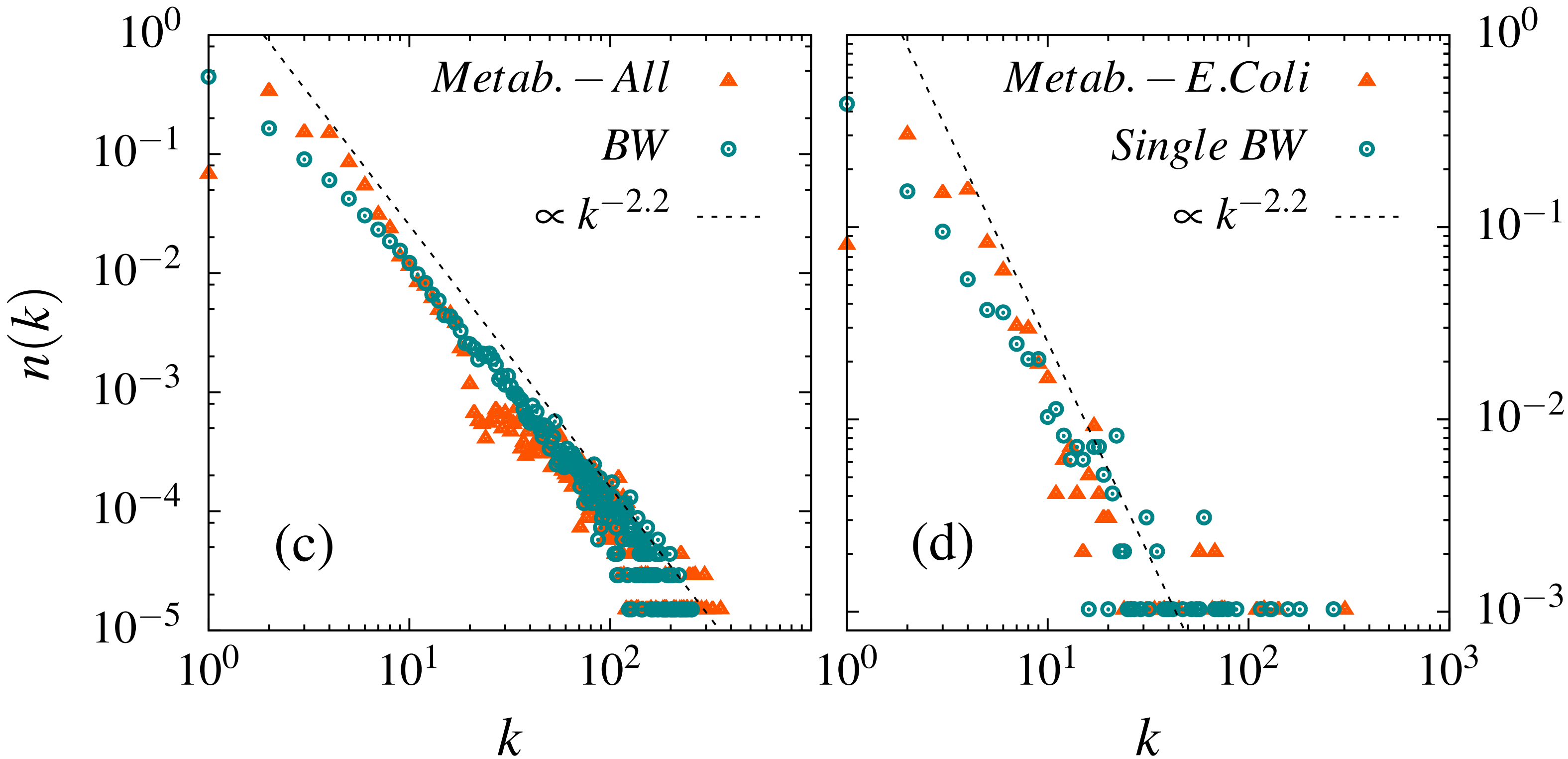
Comparison:

(c) BW vs Metabolic: same N and M

(d) BW vs E.Coli: same N and M



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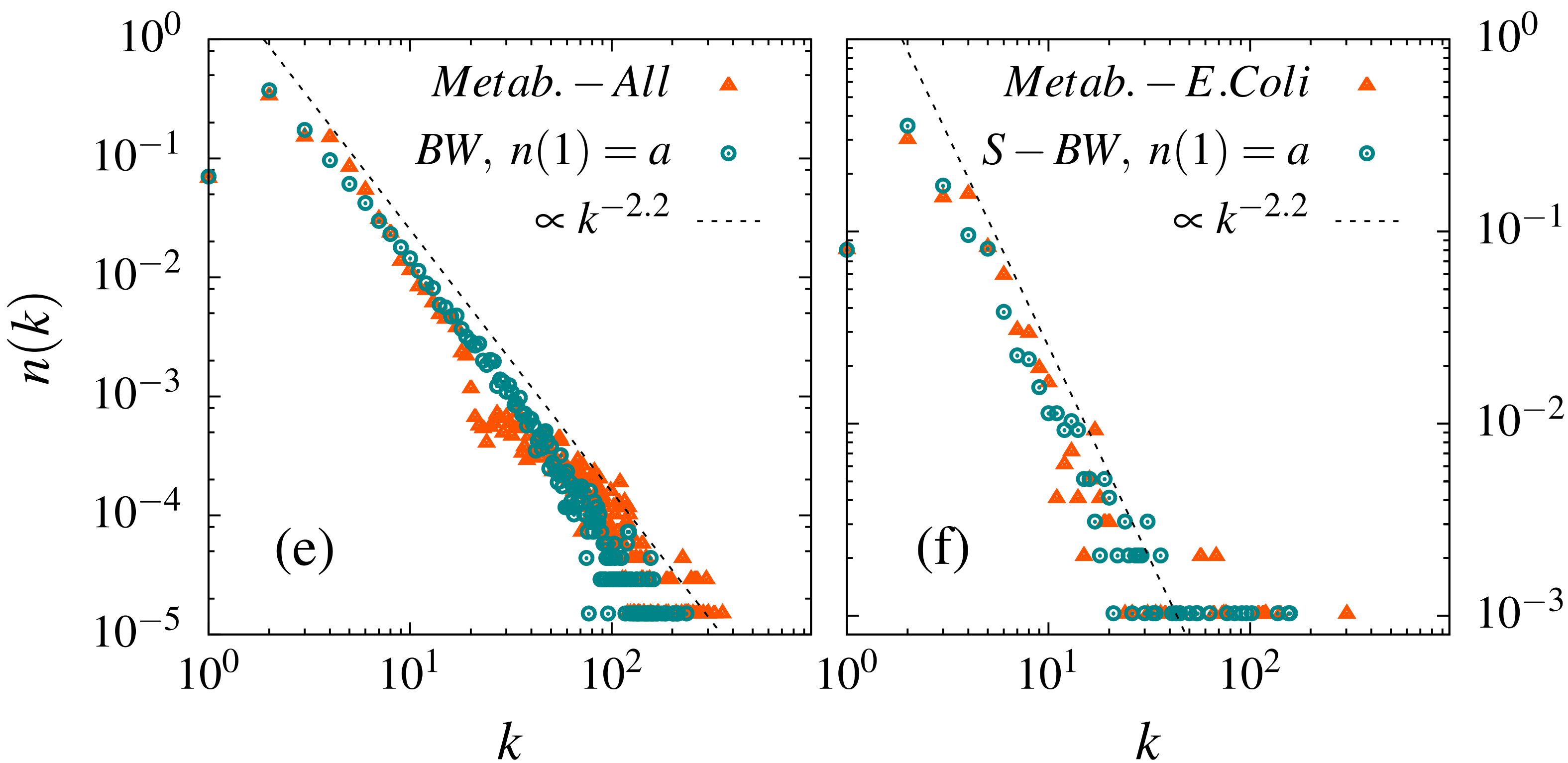
Comparison, with extra constraint:

(e) Same as in c) but with fixed n(1)

(f) Same as in d) but with fixed n(1)



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PLoS ONE 3(2): e1690,(2008).

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- BW is a random network just as ER is a random network.

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# Network properties

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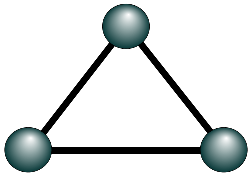
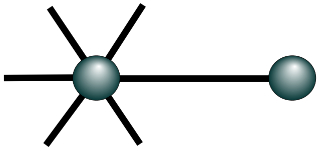
## Clustering Coefficient



$$0 \leqslant C \leqslant 1$$

## Assortativity



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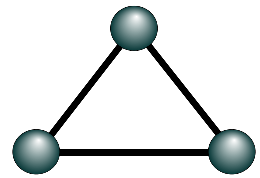

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$$\langle C \rangle_{metab} = 0.139\ (0.143)$$

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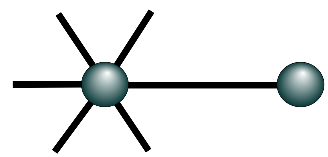

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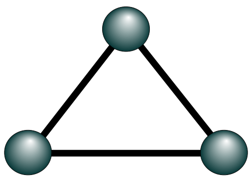

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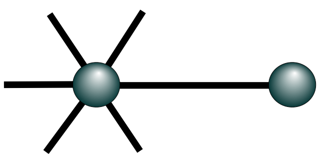
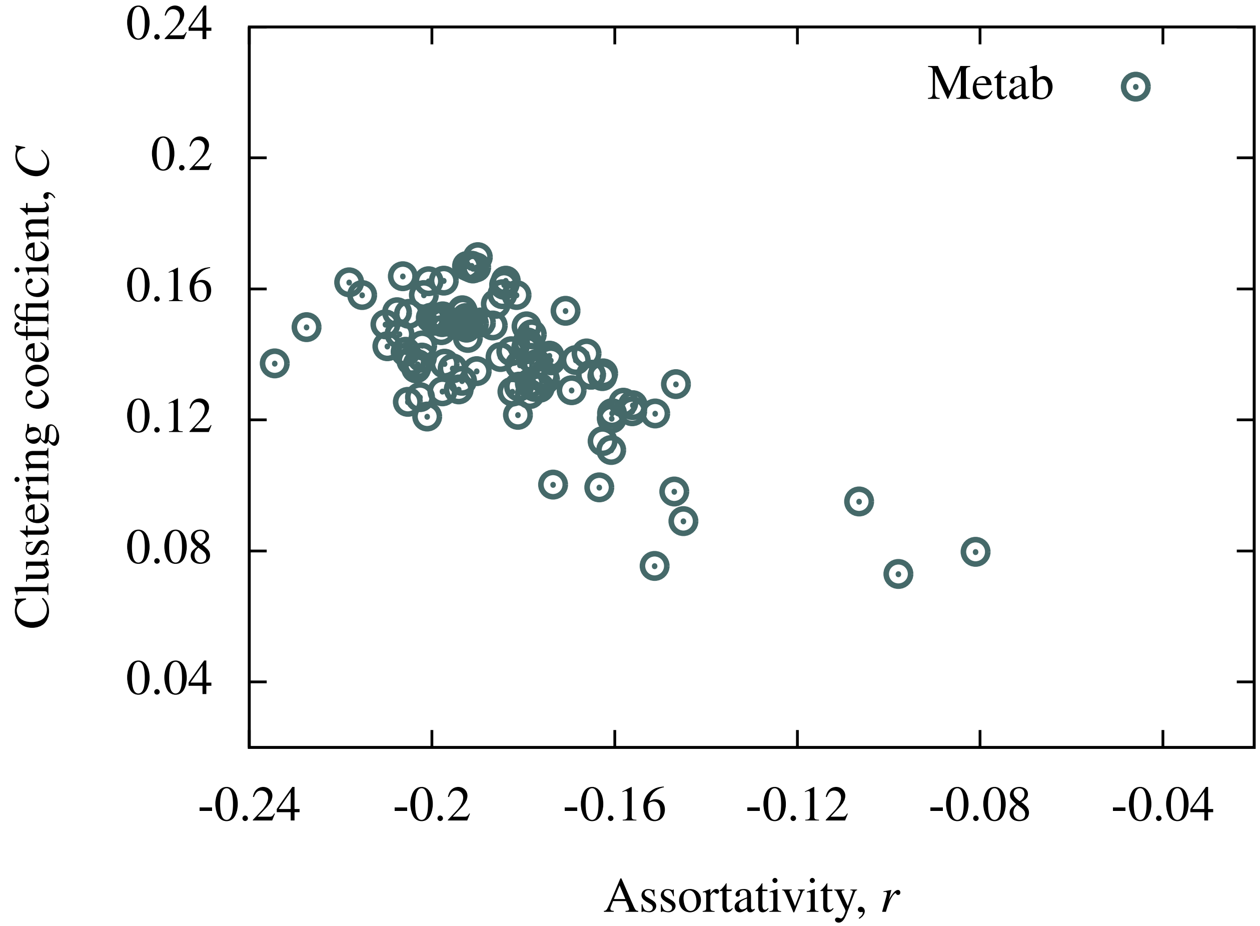


$$-1 \leqslant r \leqslant 1$$

$$\langle r \rangle_{metab} = -0.18\ (-0.178)$$

$$\langle r \rangle_{BW} = -0.123\ (-0.125)$$

# Clustering-Assortativity space



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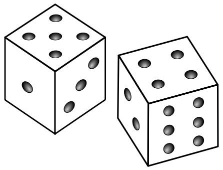
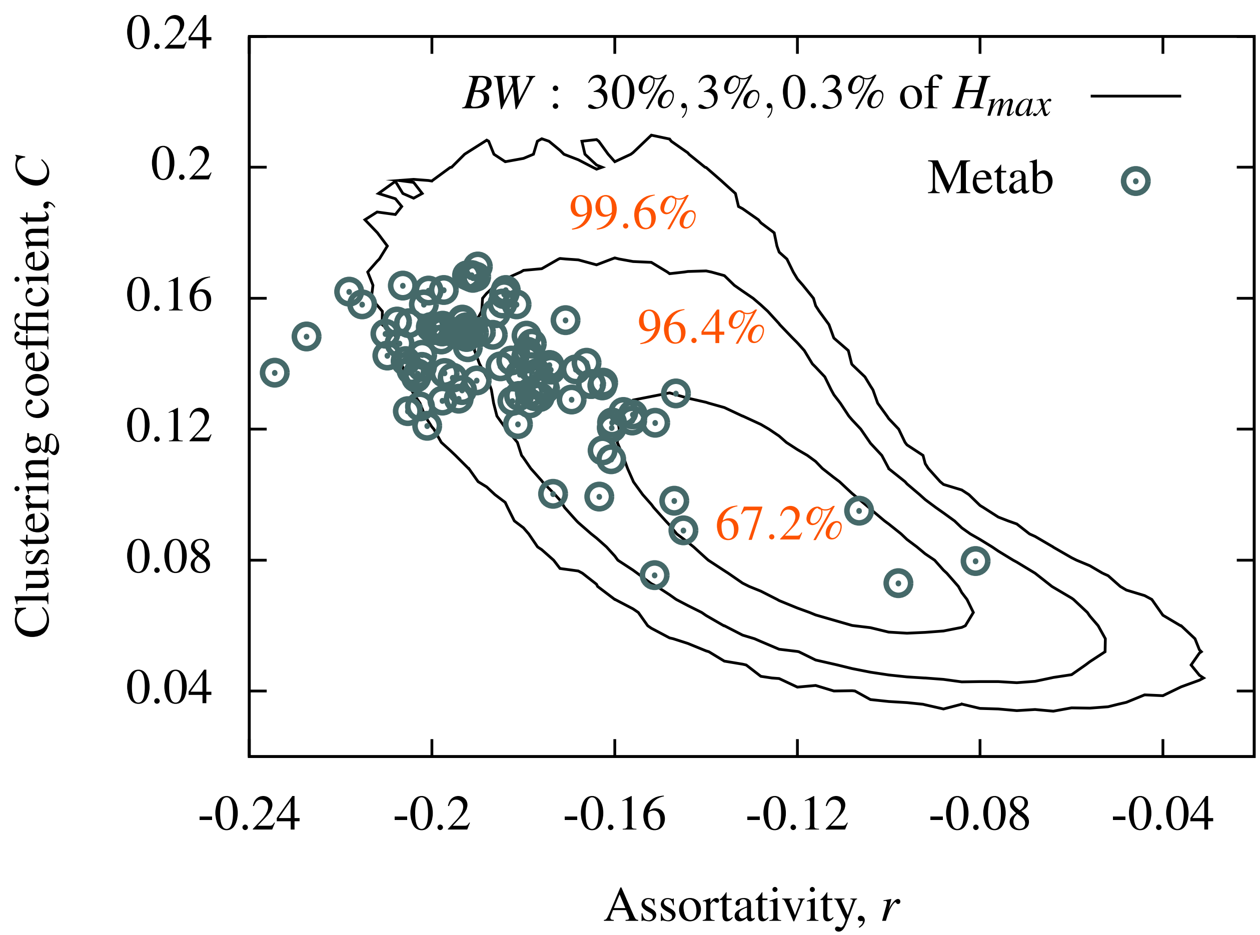
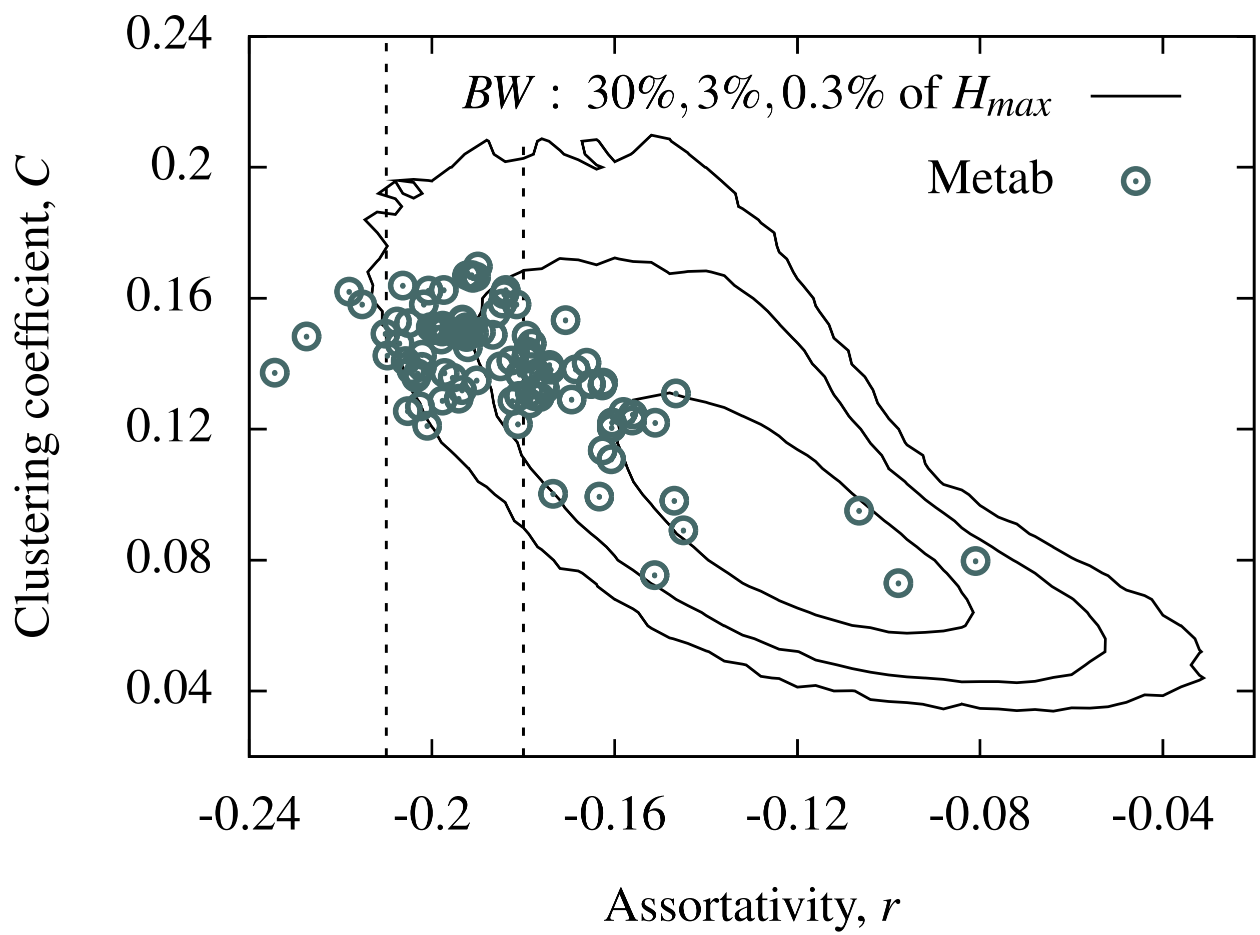




# Region of Low Assortativity

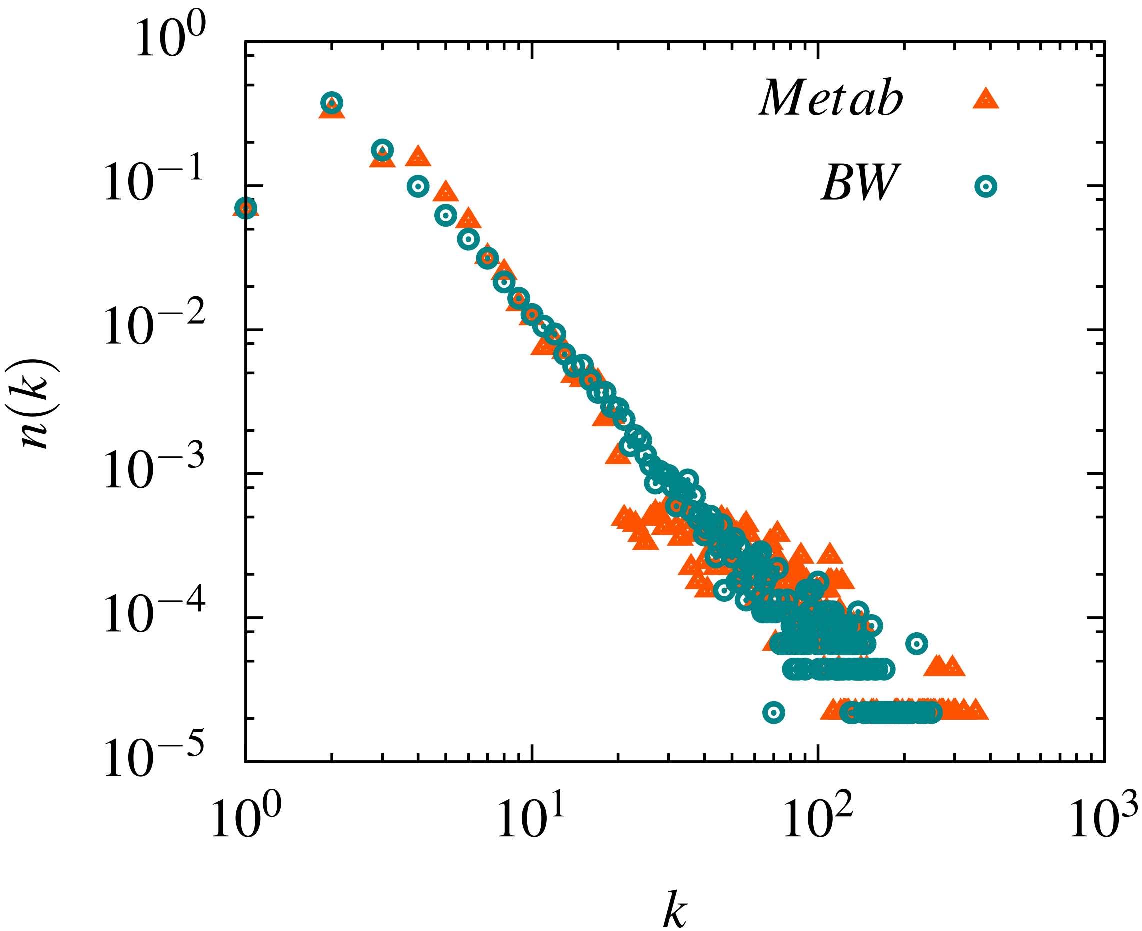



$$-0.21 < r < -0.18 \Rightarrow N_{metab} = 62$$

# Region of Low Assortativity

$\langle C \rangle_{metab} = 0.148$

$\langle C \rangle_{BW} = 0.149$



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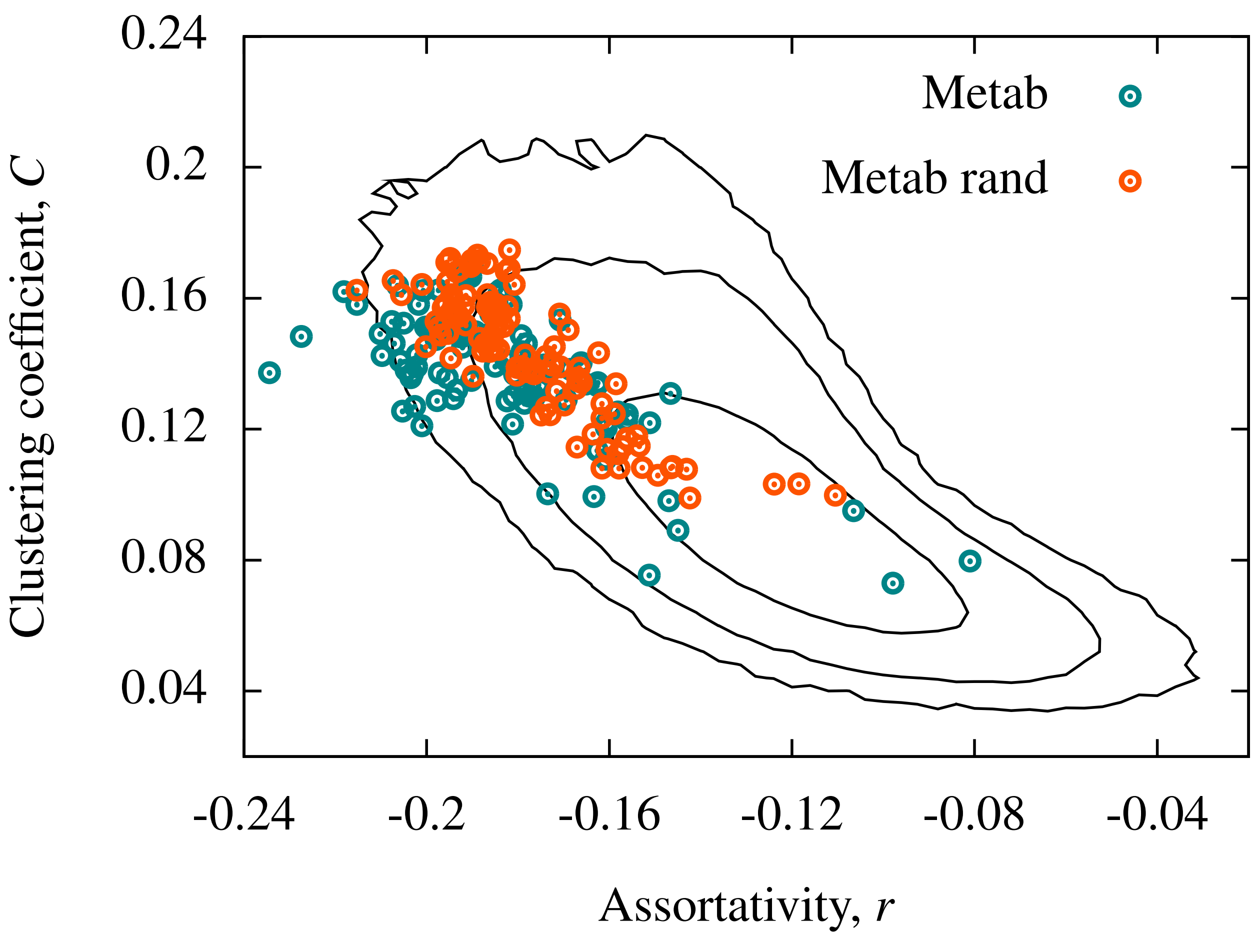
- This pressure, to large extent, is reflected in a small change in the degree distribution.
- No evolutionary pressure on clustering.

# Acknowledgment

I would like to thank...

- The organizers for inviting me.
- Petter Minnhagen for the collaboration on this work.
- Luis Rocha and Seung Ki Baek for comments and suggestions.


Metab
Metab rand
Clustering coefficient, C
Assortativity, r
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