Universality of Jamming Criticality in Overdamped Shear-Driven Frictionless Disks
Supplemental Material

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TRANSVERSE VELOCITY CORRELATION FUNCTION

The one quantity for which models RD_0 and CD_0 are clearly different is the transverse velocity correlation function, $g_y(x) \equiv \langle v_y(0)v_y(x)\rangle$. Defining the normalized correlation, $G_y(x) \equiv g_y(x)/g_y(0)$, we plot in Fig. 1(a) $G_y(x)$ vs x, for several different values of strain rate $\dot\gamma$, for model RD_0 at $\phi = 0.8433 \approx \phi_J$ in a system of $N = 4096$ particles. We see that $G_y(x)$ has a clear minimum at a distance $x = \ell$, and that ℓ increases as $\dot\gamma \to 0$ and one approaches the critical point. In Ref. [1] ℓ was interpreted as the diverging correlation length ξ. In CD_0 however, it was found [2] that $G_y(x)$ decreases monotonically without any obvious strong dependence on either ϕ or $\dot\gamma$. In Fig. 1(b) we plot $G_y(x)$ vs x, for several different $\dot\gamma$, at $\phi = 0.8433 \approx \phi_J$ in a system of $N = 4096$ particles, confirming this result.

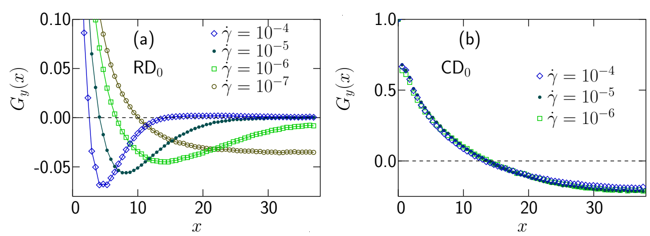

FIG. 1. Normalized transverse velocity correlation function $G_y(x) = g_y(x)/g_y(0)$ at $\phi = 0.8433 \approx \phi_J$ for a system of $N = 4096$ particles. Panel (a) is for model RD_0 with shear rates $\dot\gamma = 10^{-7}$ through 10^{-4}. Panel (b) for model CD_0 at shear rates $\dot\gamma = 10^{-6}$, through 10^{-4}.

As an alternative way to consider the difference in this correlation between the two models, we now consider the Fourier transformed correlation $g_y(k_x) = \int dx\, g_y(x)\mathrm{e}^{ik_x x}$, which we show in Figs. 2(a) and 2(b) for RD_0 and CD_0 respectively at packing fraction $\phi = 0.8433 \approx \phi_J$. For RD_0 we see a maximum in $g_y(k_x)$ at a k_x^* that decreases for decreasing $\dot\gamma$; $\ell \sim 1/k_x^*$ gives the corresponding minimum of the real-space correlation. For CD_0 we show results only for the single strain rate $\dot\gamma = 10^{-6}$ since from Fig. 1(a) we expect no observable difference as $\dot\gamma$ varies. We see an algebraic divergence $g_y(k_x) \sim k_x^{-1.3}$ as $k_x \to 0$. It is this algebraic divergence that causes the real space $G_y(x)$ in CD_0 to become solely a function of x/L as the system length L increases, as was

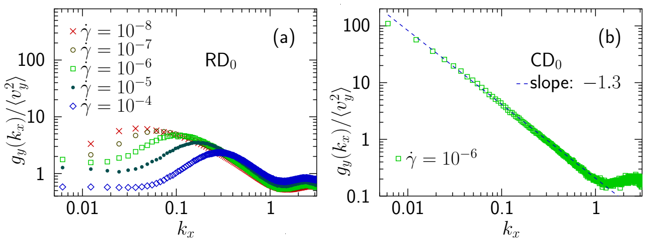

FIG. 2. Fourier transform of the transverse velocity correlation function $g_y(k_x)$ at $\phi = 0.8433 \approx \phi_J$. Panel (a) is for model RD_0 with shear rates $\dot\gamma = 10^{-8}$ through 10^{-5}. The peak in $g_y(k_x)$, moving to smaller k_x as $\dot\gamma$ decreases, is related to the minimum in the real space $g_y(x)$ moving to larger x. The algebraic behavior in panel (b) for model CD_0 at $\dot\gamma = 10^{-6}$, is consistent with the absence of any apparent length scale, as reported in Ref. [2]. The number of particles in these figures are $N = 262144$ except for the two smallest shear rates for RD_0 for which $N = 65536$.

reported in Ref. [2].

To try and give a qualitative understanding of this differing behavior of $g_y(k_x)$, we can consider how energy is dissipated in each model. In RD_0 the dissipation is $(1/N)\sum_i\langle|\delta\mathbf{v}_i|^2\rangle \approx \int d\mathbf{k}\langle\delta\mathbf{v}(\mathbf{k})\cdot\delta\mathbf{v}(-\mathbf{k})\rangle$. For CD_0, however, the dissipation is $(1/N)\sum_{i,j}\langle|\mathbf{v}_i - \mathbf{v}_j|^2\rangle \approx \int d\mathbf{k}\langle\delta\mathbf{v}(\mathbf{k})\cdot\delta\mathbf{v}(-\mathbf{k})\rangle|\mathbf{k}|^2$, where the sum is over only neighboring particles i, j in contact. Here $\delta\mathbf{v}$ is the non-affine part of the particle velocity. If we make an equipartition-like ansatz, and assume that as $k \to 0$ all modes $\mathbf{k}$, and both spatial directions x, y, contribute equally to the dissipation, we would then conclude that for RD_0 $\langle v_y(\mathbf{k})v_y(-\mathbf{k})\rangle \propto$ constant, while for CD_0 $\langle v_y(\mathbf{k})v_y(-\mathbf{k})\rangle \propto 1/k^2$. Noting that $g_y(k_x) = \int dk_y\langle v_y(\mathbf{k})v_y(-\mathbf{k})\rangle$, we then conclude that for RD_0 we have $g(k_x) \propto$ constant as $k_x \to 0$, while for CD_0 we have the divergence $g(k_x) \propto 1/k_x$. This saturation of $g_y(k_x)$ for RD_0, as compared to the algebraic divergence of $g_y(k_x)$ for CD_0, is what is qualitatively seen in Fig. 2.

The physical reason for this dramatic difference can be viewed as follows. For CD_0, since the dissipation depends only on velocity differences, uniform translation of a large cluster of particles with respect to the ensemble average flow has little cost, thus enabling long wavelength fluctuations. For RD_0 the dissipation is with respect to a fixed background, so uniform translation of a large cluster causes dissipation that scales with the cluster size; long wavelength fluctuations are suppressed.

That the observed divergence in CD_0 is $\sim k_x^{-1.3}$ rather than the simple k_x^{-1} predicted above, suggests that our equipartition ansatz is not quite correct, and that the different modes interact in a non-trivial way. That the exponent of this divergence is not an integer or simple rational fraction suggests the signature of underlying critical fluctuations, even though the correlation $g_y(x)$ itself does not yield any obvious diverging length scale.

FINITE-SIZE-SCALING OF PRESSURE

In Fig. 1 of the main article we showed data for the dependence of pressure p on system size L at different strain rates $\dot{\gamma}$, at the jamming fraction $\phi_J \approx 0.8433$. We argued that these results provided evidence for a similar growing, macroscopically large, correlation length ξ in both models RD_0 and CR_0. Here we attempt a finite-size-scaling analysis of this data. We must note at the outset, however, that our earlier work [3] demonstrated that it is important to consider corrections-to-scaling to get accurate values for the exponents at criticality, and that corrections-to-scaling are in fact large at the smaller sizes L considered in Fig. 1 of the main article [4]. Since our data for $p(L)$ is not extensive enough to try a scaling analysis including corrections-to-scaling, our results based on a fit to Eq. (5) must be viewed as providing only *effective* exponents describing the data over the range of parameters considered, rather than the true exponents asymptotically close to criticality. We restate Eq. (5),

$$p(\phi_J, \dot{\gamma}, L) = L^{-y/\nu}\, \mathcal{P}(0, \dot{\gamma} L^z). \tag{1}$$

We can equivalently write the above in the form

$$p(\phi_J, \dot{\gamma}, L) = \dot{\gamma}^{y/z\nu} f(L\dot{\gamma}^{1/z}), \tag{2}$$

using $f(x) \equiv x^{-y/\nu}\, \mathcal{P}(0, x^z)$. We can now adjust the parameters $q \equiv y/z\nu$ and z to try and collapse the data to a single common scaling curve. Plotting $p/\dot{\gamma}^q$ vs $L\dot{\gamma}^{1/z}$ we show the results for RD_0 and CD_0 in Figs. 3(a) and (b). For RD_0 we find the effective exponents $z = 6.5$ and $q = 0.290$, while for CD_0 we find $z = 6.0$ and $q = 0.317$. The values of z found in the present analysis are comparable to the value $z = 5.6$ found in the cruder analysis in Fig. 1(b) of the main article. Note that for both models the scaling function $f(x) \to$ constant as $x \to \infty$, which gives $p \sim \dot{\gamma}^q$, $q \equiv y/z\nu$, in the limit of an infinite sized system.

The closeness of these fitted *effective* exponents for the two models is one more piece of evidence that RD_0 and CD_0 behave qualitatively the same, and do not have dramatically different rheology as was claimed by Tighe et al. in Ref. [2].

Finally we consider how the effective exponents found here compare to the true exponents asymptotically close to criticality. From our most accurate analysis [3] of

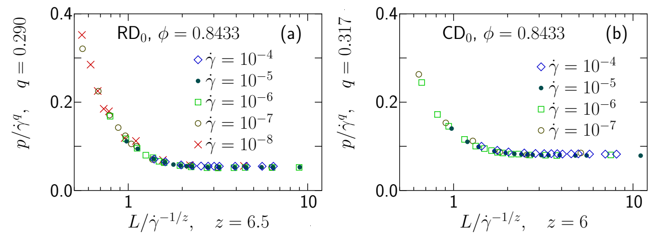

FIG. 3. Scaling collapse of pressure according to Eq. (2) for models RD_0 and CD_0.

the critical behavior in RD_0, using a large system size $N = 65536$ and including the leading corrections-to-scaling, we have found the critical exponents $q = y/z\nu = 0.28 \pm 0.02$ and $y = 1.08 \pm 0.03$, yielding $z\nu = 3.9 \pm 0.4$. This value of q is in reasonable agreement with that found above from the finite-size-scaling analysis of $p(\phi_J, \dot{\gamma}, L)$. If we take the value of $z \approx 6$ found in the finite-size-scaling analysis, we would then conclude $\nu \approx 0.65$. We note that earlier scaling analyses [1, 5] that similarly ignored corrections-to-scaling found similar values for ν. However our recent [4] more detailed finite-size-scaling analysis of the correlation length exponent, which included corrections-to-scaling, found that $\nu \approx 1$, therefore implying $z \approx 3.9$ as the true critical value. We thus conclude that the larger than expected value of z found here from the finite-size-scaling of p is due to the strong corrections-to-scaling that effect the correlation length at small L.

As another way to see the effect of corrections-to-scaling on the correlation length, in Fig. 4 we plot our results for p vs L at $\phi = 0.8433 \approx \phi_J$, as obtained from quasistatic simulations [4, 6] representing the $\dot{\gamma} \to 0$ limit. From Eq. (1) we expect as $\dot{\gamma} \to 0$ the behavior, $p \sim L^{-y/\nu}$. If we fit the data at small L in Fig. 4 to a power law, we then find the exponent, $y/\nu \approx 1.79$. Using $y = 1.08$ this then gives $\nu \approx 0.60$, in rough agreement with the value of ν obtained from the measured z of our finite-size-scaling of p with $\dot{\gamma}$. If, however, we fit the data at only the largest L to a power law, we then find the exponent $y/\nu \approx 1.11$. Again using $y = 1.08$, we then get $\nu \approx 0.97$, in better agreement with the expected $\nu \approx 1$. Fig. 4 thus shows in a very direct way that corrections-to-scaling are significant for small system lengths L.

To conclude this section, although our finite-size-scaling of the pressure data in Fig. 1(a) of the main article is effected by corrections-to-scaling, and so gives a larger value for the dynamic exponent z than we believe is actually the case at criticality, nevertheless the correlation length ξ extracted from this data and shown in Fig. 1(b) demonstrates that RD_0 and CD_0 are behaving qualitatively the same, and that both have a macroscopic length scale ξ that is growing (and we would argue diverging) as the jamming transition is approached.

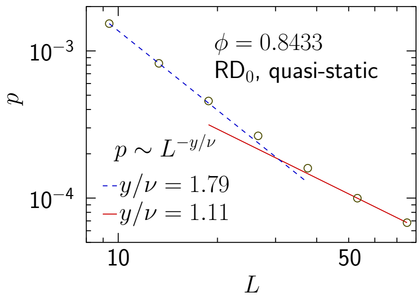

FIG. 4. Pressure p vs system length L at $\phi_J \approx 0.8433$ for quasistatic shearing. Dashed line is a power law fit to the data at the smallest L, giving an exponent $y/\nu \approx 1.79$; solid line is a power law fit to the data at the largest L, giving an exponent $y/\nu \approx 1.11$.

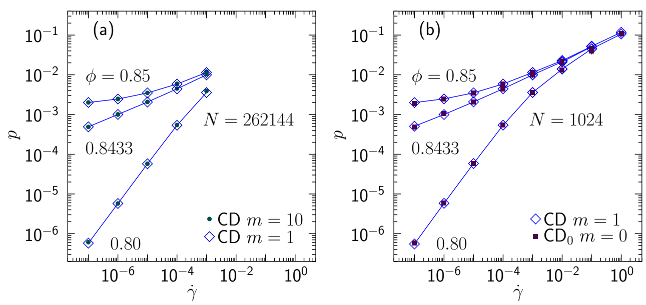

FIG. 5. Pressure p vs. shear strain rate $\dot\gamma$ at packing fractions $\phi = 0.80$, 0.8433, 0.85 for: (a) model CD with $m = 1$ and $m = 10$ for $N = 262144$ particles, and (b) model CD with $m = 1$ and model CD$_0$ with $m = 0$ for $N = 1024$ particles.

EFFECT OF FINITE MASS ON MODEL CD

We wish to verify that the mass parameter $m = 1$, which we use in model CD, is indeed sufficiently small so as to put our results in the overdamped $m \to 0$ limit corresponding to model CD$_0$, for the range of parameters studied here. In Fig. 5(a) we show results for the elastic part of the pressure p vs $\dot\gamma$ for model CD, with $N = 262144$ particles, at three different packing fractions: $\phi = 0.80$, $\phi = 0.8433 \approx \phi_J$, and $\phi = 0.85$. We compare results for two different mass parameters, $m = 1$ and $m = 10$. We see that the results agree perfectly for small $\dot\gamma$; significant differences are only found for $\dot\gamma \geq 10^{-3}$ which is higher than the largest shear rate used in our scaling analysis. In Fig. 5(b) we similarly compare results for model CD with $m = 1$ with explicit results for model CD$_0$, as obtained from simulations using the more costly matrix inversion dynamics for $m = 0$. In this case we are restricted to $N = 1024$ particles because our algorithm for CD$_0$ scales as N^2. We see that in all cases there is no observed difference between the two models. Thus we conclude that our results from CD with $m = 1$ are indeed in the overdamped $m \to 0$ limit.

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