Dynamic critical behavior of the XY model in small-world networks

Katya Medvedyeva

Department of Theoretical Physics, Umeå University, Sweden

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Ph.D. Petter Holme, *Umeå University, Sweden*
Dr. Kim Beom Jun, *Ajou University, Korea*
Prof. Petter Minnhagen, *Nordic Institute for Theoretical Physics, Denmark*

CONTENT

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- The Watts and Strogartz (WS) model of small-world networks
- XY model
- Length scales
- Static critical exponents
- Dynamic critical exponents:
  - used methods
  - finite-size scaling
- Discussion of the results

What do we study?

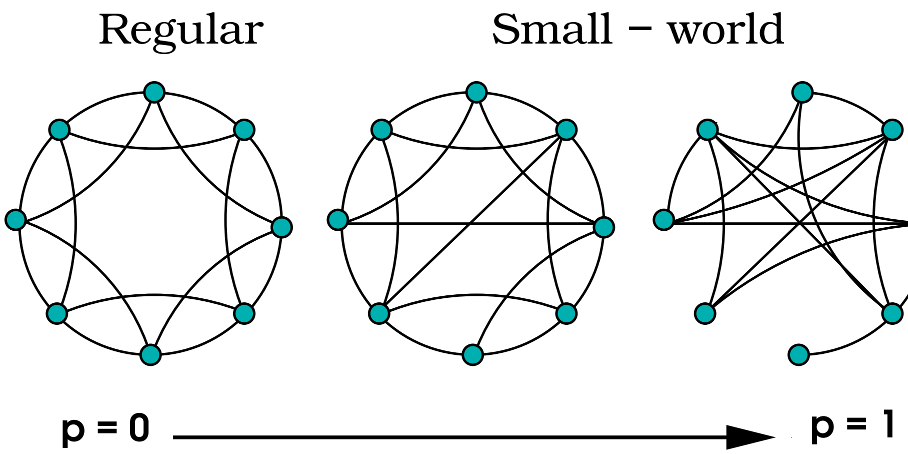
Critical behavior of the XY model on the Watts and Strogartz model of small-world networks

How do we study it?

- Monte Carlo simulations
- the short-time relaxation method
- finite-size scaling

What do we obtain?

The static and dynamic critical exponents α, β, ν, z



Parameters

N vertices

k edges per vertex

P rewiring probability

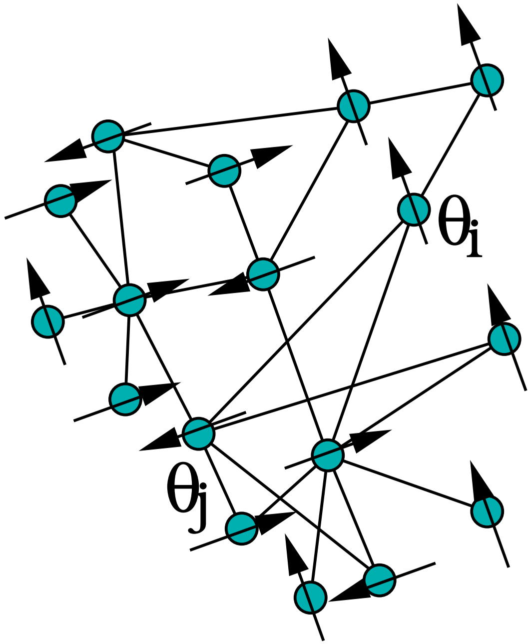
The average path length

$$\underline{l \sim N \quad \text{if} \quad P = 0} \qquad \text{and} \qquad \underline{l \sim \log N \quad \text{if} \quad P > 0}$$

$\Longrightarrow$

small-world phenomenon emerge at any finite P

D. J. Watts and S. H. Strogatz, "Collective dynamics of 'small-world' networks" *Nature* 393:440-42.



$$\Psi(\mathbf{r}) = |\Psi(\mathbf{r})| e^{i\theta(\mathbf{r})}$$

$$\Downarrow$$

$$|\Psi_i| = |\Psi| e^{i\theta_i}$$

The Hamiltonian:

$$H = -\frac{1}{2} \sum_{i \neq j} J_{ij} \cos(\theta_i - \theta_j)$$

where

$$J_{ij} = J_{ji} = \begin{cases} J, & \text{if } (i, j) \text{ is an edge,} \\ 0, & \text{otherwise.} \end{cases}$$

- Transition from disordered to ordered state at T_c for d≥ 2
- Quasi-long-range order for d$= 2$

d-dimensional regular lattice

- linear size of the system L
- the correlation length

$$\xi \sim |T - T_c|^{-\nu}$$

- the relaxation time

$$\tau \sim \xi^z = L^z$$

$$\Downarrow$$

$$\boxed{\chi(t/L^z, \xi/L)}$$

Network

- the network size $N = L^d$
- the correlation volume

$$\xi_V \sim \xi^d \quad \Rightarrow \quad \xi_V \sim |T - T_c|^{-\bar{\nu}}$$
$$\Longrightarrow \quad \bar{\nu} = d\nu$$

- the relaxation time

$$\tau \sim N^{\bar{z}} \quad \Longrightarrow \quad \bar{z} = z/d$$

- the typical distance between the ends of shortcuts

$$\zeta = (kP)^{-1}$$

$$\Downarrow$$

$$\boxed{\chi(t/N^{\bar{z}}, \xi/N, \zeta/N)}$$

Static critical exponents

- static behavior
- geometry of the network: $N \gg 1/P$ or $N \gg \zeta$

$$\chi(t/N^{\bar{z}}, \xi/N, \zeta/N) \quad \Longrightarrow \quad \chi(\xi/N, 0)$$

B. J. Kim, H. Hong, P. Holme, G. S. Jeon, P. Minnhagen and M. Y. Choi, "*XY* model in small-world networks" *Phys. Rev. E* **64**, 056135 (2001).

- $m \sim (T_c - T)^\beta \quad \Longrightarrow \quad \underline{\beta = 1/2}$

- $C_v \sim |T - T_c|^{-\alpha} \quad \Longrightarrow \quad \underline{\alpha = 0}$

- $\xi_V \sim |T - T_c|^{-\bar{\nu}} \quad \Longrightarrow \quad \bar{\nu} = 2$

mean-field transition $\Rightarrow \quad d \geq 4 \quad \Rightarrow \quad \underline{\nu = \bar{\nu}/d = 1/2}$

DYNAMIC CRITICAL EXPONENT: USED METHODS

$$\tau \sim \xi^z \qquad \text{at } T_c \Longrightarrow \qquad \underline{\tau \sim L^{-z} = N^{-\bar{z}}} \quad \text{with} \quad \bar{z} = z/d$$

- Short-time relaxation method:

$$Q(t) = \left[\langle \mathrm{sgn}(\textstyle\sum_{i=1}^{N} \cos\theta_i(t))\rangle\right]$$

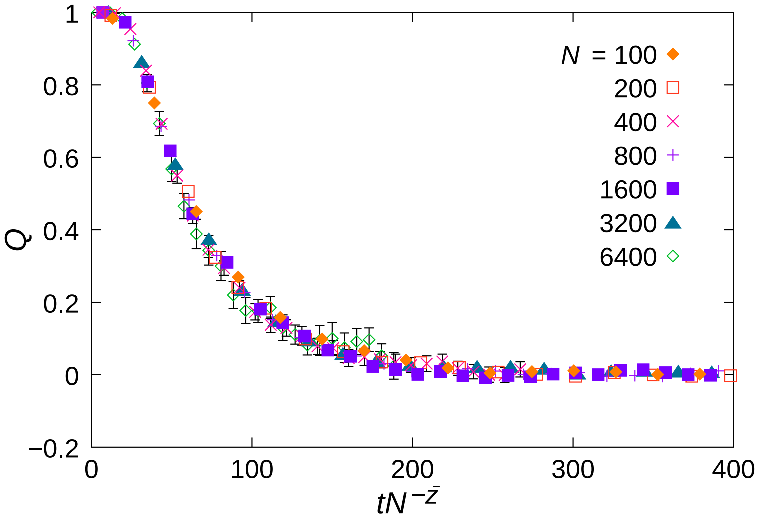
the initial configuration $\qquad \theta_i(0) = 0 \qquad \Longrightarrow \qquad Q(0) = 1$

and $\qquad Q(t \to \infty) = 0$

- Dynamic Monte-Carlo simulations

DYNAMIC CRITICAL EXPONENT: FINITE-SIZE SCALING

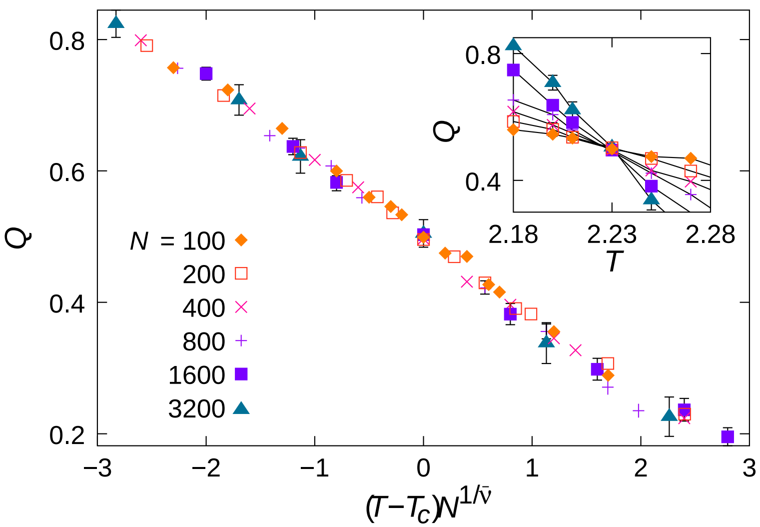
$k = 3, P = 0.2$



$$Q(t, T, N) = F(t/N^{\bar{z}}, (T - T_c)N^{1/\bar{\nu}})$$

at T_c: $\quad Q(t, N) = F(t/N^{\bar{z}})$

$$\Longrightarrow \quad T_c = 2.23, \quad \bar{z} = 0.52(1)$$

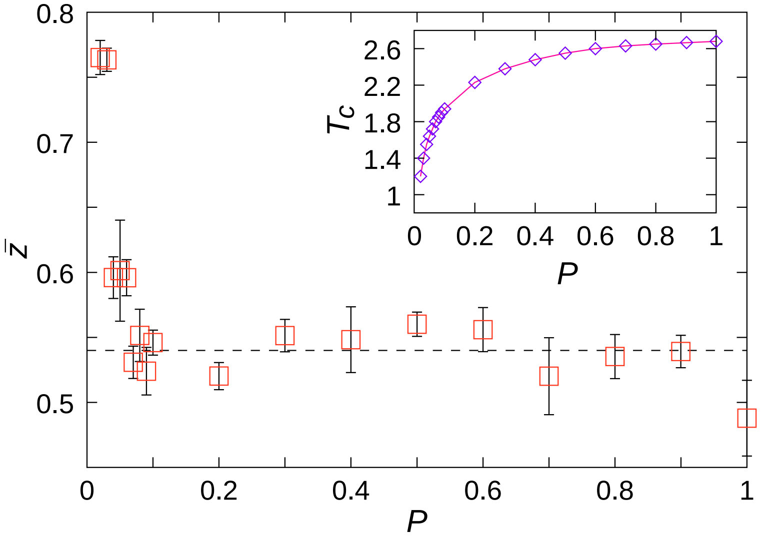


$$a = tN^{-\bar{z}} = 3$$

$$Q_a(T, N) = F(a, (T - T_c)N^{1/\bar{\nu}})$$

$$\Longrightarrow \quad T_c = 2.23, \quad \bar{\nu} = 2.0$$

DYNAMIC CRITICAL EXPONENT: DEPENDENCE FROM THE PARAMETER P



$\Longrightarrow$

- $\bar{z} \approx 0.54(3)$ for $P \gtrsim 0.03$

- $N \gg 1/P$ fails for $P \lesssim 0.03$

$\Longrightarrow$ $\boxed{\chi(t/N^{\bar{z}}, \xi/N, \zeta/N)}$

XY model on the WS model of small-world networks

- for finite P:

  $\beta = 1/2, \quad \alpha = 0, \quad \bar{\nu} = 2 \rightarrow \nu = \bar{\nu}/d = 1/2$ - <u>mean-field values</u>

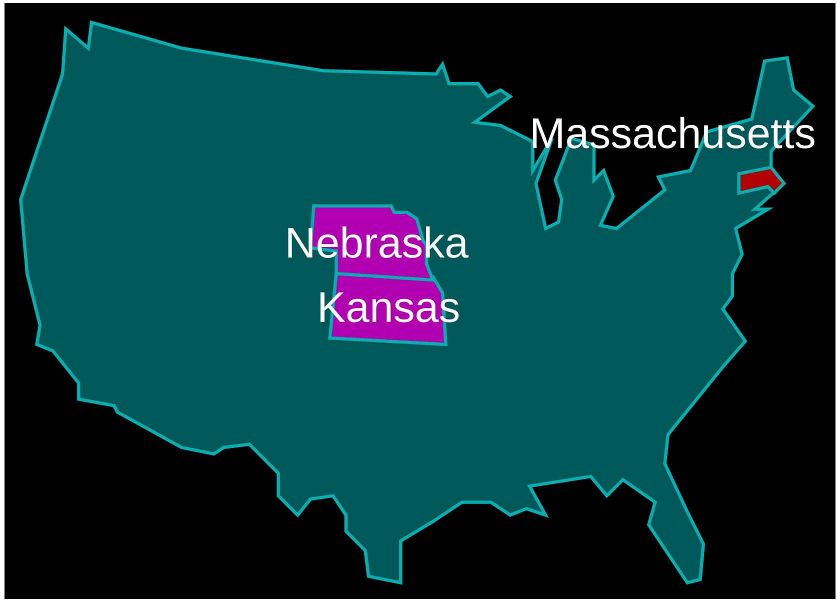
  $\bar{z} = 0.54(3) \quad \rightarrow \quad z = \bar{z}d \approx 2.1$

  $\Longrightarrow$ the same universality class as a regular lattice of $d \geq 4$

- there is no critical behavior *induced* by WS model other than the transition from large-world regime $l \sim N$ to small-world regime $l \sim \log N$

- hyper-cubic lattice: number of edges $> 8N$

- we need: $kN = 3N$ and most probably $2N$



The experiment:

- How many intermediates between people in Nebraska/Kansas and Boston.
- Target known by name and profession.
- Packages sent closer in geographical and social space to persons at least known by their first name.

The result:

- A median of 5 intermediates.
- Comparing with similar experiments on smaller populations $\Rightarrow$ a logarithmically increasing shortest path length.
- Average shortest path length of the Earth's population = 6.

S. Milgram, "The Small-World Problem" *Psychology Today* 2:60-67.