Dynamic critical behavior of the XY model in small-world networks

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Introduction

The Watts and Strogatz (WS) model of small-world networks

XY model

Length scales

Static critical exponents

Dynamic critical exponents:
  - used methods
  - finite-size scaling

Discussion of the results

http://www.tp.umu.se/~medv/
Introduction

What do we study?

Critical behavior of the XY model on the Watts and Strogatz model of small-world networks

How do we study it?

- Monte Carlo simulations
- the short-time relaxation method
- finite-size scaling

What do we obtain?

The static and dynamic critical exponents $\alpha, \beta, \nu, z$
The Watts-Strogatz Model of Small-World Network

The average path length

\[ l \sim N \quad \text{if} \quad P = 0 \quad \text{and} \quad l \sim \log N \quad \text{if} \quad P > 0 \]

⇒ small-world phenomenon emerge at any finite \( P \)

The Hamiltonian:

\[ H = -\frac{1}{2} \sum_{i \neq j} J_{ij} \cos(\theta_i - \theta_j) \]

where

\[ J_{ij} = J_{ji} = \begin{cases} J, & \text{if } (i, j) \text{ is an edge,} \\ 0, & \text{otherwise.} \end{cases} \]

- Transition from disordered to ordered state at \( T_c \) for \( d \geq 2 \)
- Quasi-long-range order for \( d = 2 \)
**LENGTH SCALES**

*d-dimensional regular lattice*

- linear size of the system \( L \)
- the correlation length \( \xi \sim |T - T_c|^{-\nu} \)
- the relaxation time \( \tau \sim \xi^z = L^z \)

\[ \downarrow \]

\[ \chi(t/L^z, \xi/L) \]

*Network*

- the network size \( N = L^d \)
- the correlation volume \( \xi_V \sim \xi^d \Rightarrow \xi_V \sim |T - T_c|^{-\bar{\nu}} \Rightarrow \bar{\nu} = d\nu \)
- the relaxation time \( \tau \sim N^\bar{z} \Rightarrow \bar{z} = z/d \)
- the typical distance between the ends of shortcuts \( \zeta = (kP)^{-1} \)

\[ \downarrow \]

\[ \chi(t/N^\bar{z}, \xi/N, \zeta/N) \]
Static critical exponents

- static behavior
- geometry of the network: \( N \gg 1/P \) or \( N \gg \zeta \)

\[
\chi(t/N^z, \xi/N, \zeta/N) \Rightarrow \chi(\xi/N, 0)
\]


- \( m \sim (T_c - T)^\beta \) \( \Rightarrow \beta = 1/2 \)
- \( C_v \sim |T - T_c|^{-\alpha} \) \( \Rightarrow \alpha = 0 \)
- \( \xi_V \sim |T - T_c|^{-\bar{\nu}} \) \( \Rightarrow \bar{\nu} = 2 \)

mean-field transition \( \Rightarrow d \geq 4 \) \( \Rightarrow \nu = \bar{\nu}/d = 1/2 \)
**Dynamic Critical Exponent: Used Methods**

\[ \tau \sim \xi^z \quad \text{at } T_c \implies \tau \sim L^{-z} = N^{-\bar{z}} \quad \text{with} \quad \bar{z} = z/d \]

- **Short-time relaxation method:**

\[
Q(t) = \langle \text{sgn}\left(\sum_{i=1}^{N} \cos \theta_i(t)\right) \rangle
\]

the initial configuration \( \theta_i(0) = 0 \implies Q(0) = 1 \)

and \( Q(t \to \infty) = 0 \)

- **Dynamic Monte-Carlo simulations**

[http://www.tp.umu.se/~medv/]
**Dynamic critical exponent: finite-size scaling**

$k = 3, P = 0.2$

$Q(t, T, N) = F(t/N^{\tilde{z}}, (T - T_c)N^{1/\tilde{\nu}})$

at $T_c$: \[ Q(t, N) = F(t/N^{\tilde{z}}) \]

$\implies T_c = 2.23, \; \tilde{z} = 0.52(1)$

$a = tN^{-\tilde{z}} = 3$

$Q_a(T, N) = F(a, (T - T_c)N^{1/\tilde{\nu}})$

$\implies T_c = 2.23, \; \tilde{\nu} = 2.0$

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Dynamic critical exponent: dependence from the parameter $P$

\[ z \approx 0.54(3) \quad \text{for} \quad P \gtrsim 0.03 \]

\[ N \gg 1/P \quad \text{fails for} \quad P \lesssim 0.03 \]

\[ \chi(t/N^z, \xi/N, \zeta/N) \]
**Discussion of the results**

**XY model on the WS model of small-world networks**

- for finite $P$:
  
  $\beta = 1/2, \quad \alpha = 0, \quad \bar{\nu} = 2 \rightarrow \nu = \bar{\nu}/d = 1/2$  
  
  - mean-field values

  $\bar{z} = 0.54(3) \rightarrow z = \bar{z}d \approx 2.1$

  $\Rightarrow$ the same universality class as a regular lattice of $d \geq 4$

- there is no critical behavior *induced* by WS model other than the transition from large-world regime $l \sim N$ to small-world regime $l \sim \log N$

- hyper-cubic lattice: number of edges $> 8N$

- we need: $kN = 3N$ and most probably $2N$
The experiment:

- Packages sent closer in geographical and social space to persons at least known by their first name.

The result:

- A median of 5 intermediates.
- Comparing with similar experiments on smaller populations ⇒ a logarithmically increasing shortest path length.
- Average shortest path length of the Earth’s population = 6.