5.1
(5.2)
5.3

9: Single-mode quantum optics

Consider only a single mode.

No polarization or direction needs to be considered, so

$$\hat{E}(\chi) = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\left(\underbrace{\hat{a}e^{-i\chi}}_{\hat{E}^+} + \underbrace{\hat{a}^+ e^{i\chi}}_{\hat{E}^-}\right)$$

where $\chi = \omega t - kz - \frac{\pi}{2}$ as before, and $\omega = ck$

Choose units so that

$$\hat{E}(\chi) = \frac{1}{2}\hat{a}e^{-i\chi} + \frac{1}{2}\hat{a}^+ e^{i\chi} = \hat{X}\cos\chi + \hat{Y}\sin\chi$$

in terms of quadrature modes.

Measure $\hat{E}$ at two different times or points $\hat{E}(\chi_1)$ and $\hat{E}(\chi_2)$

the commutator becomes

$$\left[\hat{E}(\chi_1), \hat{E}(\chi_2)\right] = -\frac{i}{2}\sin(\chi_1 - \chi_2)$$

By the Heisenberg uncertainty relation, we have:

$$\Delta E(\chi_1)\,\Delta E(\chi_2) \geq \frac{1}{4}\left|\sin(\chi_1 - \chi_2)\right|$$

for the standard deviations ΔE, where

$(\Delta E)^2 = \langle \hat{E}^2 \rangle - \langle \hat{E} \rangle^2$ variance.

∴ We cannot measure precisely the values of $\hat{E}$ at both points

– unless χ_1 and χ_2 differ by multiple of π.

9-2 <u>Ability of the light wave to carry information</u>

Define the signal-to-noise ratio:

$$\boxed{SNR = \frac{\text{signal}}{\text{noise}} = \frac{\langle E(x)\rangle^2}{(\Delta E(x))^2}}$$

expectation value in some state.

<u>Coherences</u>

$$\boxed{g^{(1)}(z_1 t_1; z_2 t_2)} = g^{(1)}(x_1, x_2) =$$

$$\frac{\langle E^-(x_1)E^+(x_2)\rangle}{\langle E^-(x_1)E^+(x_1)\rangle} = \frac{\langle \hat{a}^+ e^{ix_1}\hat{a}e^{-ix_2}\rangle}{\langle \hat{a}^+ e^{ix_2}\hat{a}e^{-ix_1}\rangle} = \boxed{e^{i(x_1-x_2)}}$$

$$\boxed{g^{(2)}(x_1 x_2)} = \frac{\langle E^-(x_1)E^-(x_2)E^+(x_2)E^+(x_1)\rangle}{\langle E^-(x_1)E^+(x_1)\rangle\langle E^-(x_2)E^+(x_2)\rangle}$$

use $[a, a^+]=1$ and $\hat{a}^+\hat{a}=\hat{n}$

$$= \frac{\langle a^+a^+aa\rangle}{\langle a^+a\rangle\langle a^+a\rangle} = \frac{\langle a^+(aa^+-1)\hat{a}\rangle}{\langle a^+a\rangle^2} = \frac{\langle \hat{n}^2-\hat{n}\rangle}{\langle n\rangle^2}$$

$$= \frac{\langle \hat{n}^2\rangle - \langle\hat{n}\rangle^2 + \langle\hat{n}\rangle^2 - \langle\hat{n}\rangle}{\langle\hat{n}\rangle^2} = \boxed{1 + \frac{(\Delta n)^2 - \langle n\rangle}{\langle n\rangle^2}}$$

for <u>single-mode light</u>.

<u>Note</u> that $1 - \frac{1}{\langle n\rangle} \leq g^{(2)}(\tau)$ if $\langle n\rangle \geq 1$

0 if $\langle n\rangle \leq 1$

instead of the classical lower limit 0.

Single-mode number states

$$\hat{n}|n\rangle = n|n\rangle \implies \hat{n}^2|n\rangle = n^2|n\rangle$$

$$\implies (\Delta n)^2 = \langle n^2\rangle - \langle n\rangle^2 = 0$$

and $g^{(2)}(\tau) = 1 - \frac{1}{n}$ for $n \geq 1$

In terms of quadratures, have

$$\hat{n} + \frac{1}{2} = \hat{a}^{+}\hat{a} + \frac{1}{2} = \hat{X}^2 + \hat{Y}^2$$

Expectation values are

$$\langle n|\hat{X}|n\rangle = \langle n|\hat{Y}|n\rangle = 0$$

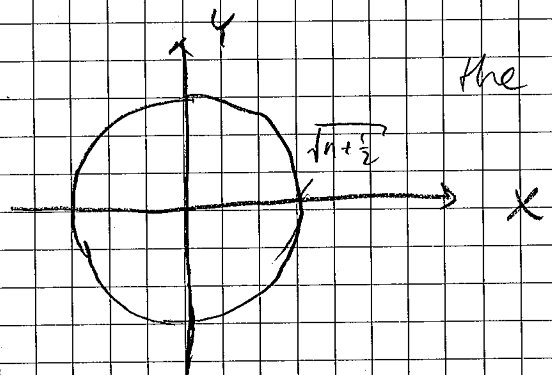
$$(\Delta X)^2 = \langle n|\hat{X}^2|n\rangle = \langle n|\frac{1}{4}(\hat{a}+\hat{a}^{+})(\hat{a}+\hat{a}^{+})|n\rangle$$

$$= \frac{1}{4}\langle n|\underbrace{\hat{a}\hat{a}}_{0} + \underbrace{\hat{a}^{+}\hat{a}^{+}}_{0} + \hat{a}^{+}\hat{a} + \hat{a}\hat{a}^{+}|n\rangle$$

$$= \frac{1}{2}\left(n + \frac{1}{2}\right)$$

$$(\Delta Y)^2 = \frac{1}{2}\left(n + \frac{1}{2}\right)$$

Represent this in a X-Y-diagram as



the possible values of X, Y

9-4) E-field

$$\langle n|\hat{E}(x)|n\rangle = 0$$

$$(\Delta E(x))^2 = \frac{1}{2}(n+\tfrac{1}{2})$$

→ a wave with amplitude $E_0 = \sqrt{n+\frac{1}{2}}$ but completely uncertain phase φ.

Coherent states

A coherent state is defined by

$$\boxed{|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle}$$

Alternative way to write it

using $|n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n |0\rangle$

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{(\alpha \hat{a}^\dagger)^n}{n!} |0\rangle$$

$$\boxed{|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2 + \alpha \hat{a}^\dagger} |0\rangle}$$

where the meaning of $e^{\alpha \hat{a}^\dagger}$ is the Taylor expansion.

Note that

$$\boxed{\hat{a}|\alpha\rangle} = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=1}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \sqrt{n}\, |n-1\rangle =$$

$$= e^{-\frac{1}{2}|\alpha|^2} \sum_{n=1}^{\infty} \alpha \cdot \frac{\alpha^{n-1}}{\sqrt{(n-1)!}} |n-1\rangle$$

$$= \alpha\, e^{-\frac{1}{2}|\alpha|^2} \sum_{n=1}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$= \boxed{\alpha |\alpha\rangle}$$

The coherent state is an eigenstate of $\hat{a}$!

It also holds that

$$|\alpha\rangle = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} |0\rangle \equiv \hat{D}(\alpha) |0\rangle$$

9-6 Calculate some properties

- $\langle \hat{n} \rangle = \langle \alpha | \hat{a}^+ \hat{a} | \alpha \rangle = \langle \alpha | \alpha^* \alpha | \alpha \rangle = |\alpha|^2$

$$\hat{n}^2 = \hat{a}^+ \hat{a} \hat{a}^+ \hat{a} = \hat{a}^+ (\hat{a}^+ \hat{a} + 1) \hat{a} = \hat{a}^+ \hat{a}^+ \hat{a} \hat{a} + \hat{a}^+ \hat{a}$$

$$\Rightarrow \langle \hat{n}^2 \rangle = \langle \alpha | \hat{a}^+ \hat{a}^+ \hat{a} \hat{a} | \alpha \rangle + \langle \alpha | \hat{a}^+ \hat{a} | \alpha \rangle = |\alpha|^4 + |\alpha|^2$$

$= \langle n \rangle^2 + \langle n \rangle$ in this case

so that

$$(\Delta n)^2 = |\alpha|^2 = \langle n \rangle$$

$\Rightarrow \dfrac{\Delta n}{n} = \dfrac{1}{\sqrt{\langle n \rangle}}$ for a coherent state

Degree of coherence

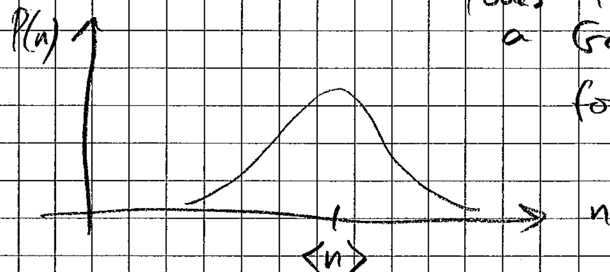
$$\boxed{g^{(2)}(\tau)} = 1 + \frac{(\Delta n)^2 - \langle n \rangle}{\langle n \rangle^2}$$ for single-mode light in general

$$= \boxed{1}$$ for coh. states

- What is the probability of finding n photons in a coherent state in a measurement?

$$\boxed{P(n)} = |\langle n | \alpha \rangle|^2 = \left| e^{-|\alpha|^2/2} \frac{\alpha^n}{n!} \right|^2 = \boxed{e^{-\langle n \rangle} \frac{\langle n \rangle^n}{n!}}$$

a Poisson distribution



looks like a Gaussian for $\langle n \rangle$ large

Fig. 5.3

Expectation values of quadratures

$$\langle\alpha|\hat{X}|\alpha\rangle = \frac{1}{2}\langle\alpha|\hat{a}^\dagger + \hat{a}|\alpha\rangle = \frac{1}{2}(\alpha^* + \alpha)$$

$$= \mathrm{Re}\,\alpha = |\alpha|\cos\theta$$

if $\theta = \arg(\alpha)$.

$$\langle\alpha|\hat{Y}|\alpha\rangle = \mathrm{Im}\,\alpha = |\alpha|\sin\theta.$$

Lower bound!

Variances

$$(\Delta X)^2 = (\Delta Y)^2 = \frac{1}{4} \quad \text{coherent state}$$

(cf. $(\Delta X)^2 = (\Delta Y)^2 = \frac{1}{2}(n + \frac{1}{2})$ number state)

*Expectation of E-field

$$\langle\alpha|\hat{E}(\chi)|\alpha\rangle = \frac{1}{2}\langle\alpha|\hat{a}^\dagger e^{i\chi} + \hat{a}e^{-i\chi}|\alpha\rangle$$

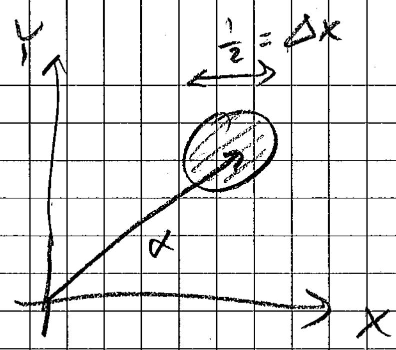
$$= \frac{1}{2}|\alpha|\left(e^{i(\chi-\theta)} + e^{i(\theta-\chi)}\right) = |\alpha|\cos(\chi-\theta)$$

$$(\Delta E(\chi))^2 = \frac{1}{4}$$

$$\therefore \boxed{SNR = 4|\alpha|\cos(\chi-\theta) = 4\langle n\rangle\cos(\chi-\theta)}$$

for coherent state.

9-8



The coherent state has slight amplitude nois and phase noise
- gets less significant for lager $\langle n \rangle = |\alpha|^2$