7. Quantization, II

4.4
4.8

Recall the energy

$$E_R = \sum_{k\lambda} \varepsilon_0 V \omega_k^2 \left(A_{\vec{k}\lambda} A^*_{\vec{k}\lambda} + A^*_{\vec{k}\lambda} A_{\vec{k}\lambda} \right)$$

Dimensions $[A_{k\lambda}] = \left[\frac{\text{energy}}{\text{volume} \cdot \text{frequency} \cdot As/Vm} \right]$

$$= \left[\frac{M}{L \cdot T \cdot (As/Vm)} \right]$$

Define $a_{k\lambda}$ so that $\begin{cases} A_{k\lambda} = a_{k\lambda} \sqrt{\frac{\hbar}{2\varepsilon_0 V \omega_k}} \\ A^*_{k\lambda} = a^*_{k\lambda} \sqrt{\frac{\hbar}{2\varepsilon_0 V \omega_k}} \end{cases}$

$\Rightarrow a_{k\lambda}$ dimensionless.

then

$$E_R = \sum_{k\lambda} \frac{\hbar \omega_k}{2} \left(a^*_{k\lambda} a_{k\lambda} + a_{k\lambda} a^*_{k\lambda} \right)$$

"Looks like h.o. Hamiltonian"

Define quantum mechanical operators $\hat{a}_{k\lambda}$, $\hat{a}^+_{k\lambda}$ and postulate (or guess!)

$$[\hat{a}_{\vec{k}\lambda}, \hat{a}^+_{\vec{k}\lambda}] = 1$$

$$[\hat{a}_{\vec{k}\lambda}, \hat{a}^+_{\vec{k}'\lambda'}] = \delta_{\vec{k}\vec{k}'} \delta_{\lambda\lambda'}$$

Then $$\hat{H}_R = \sum_{k\lambda} \frac{\hbar\omega}{2} \left(\hat{a}^+_{\vec{k}\lambda} \hat{a}_{\vec{k}\lambda} + \hat{a}_{\vec{k}\lambda} \hat{a}^+_{\vec{k}\lambda} \right)$$

$$\boxed{\hat{H}_R = \sum_{k\lambda} \hbar\omega \left(\hat{a}^+_{\vec{k}\lambda} \hat{a}_{\vec{k}\lambda} + \frac{1}{2} \right)}$$

The correctness of this is ultimately proven by experiment.

Def. number operator

$$\hat{n}_{\vec{k}\lambda} = \hat{a}^{+}_{\vec{k}\lambda}\hat{a}_{\vec{k}\lambda}$$

Eigenstates

$$\hat{n}_{\vec{k}\lambda}|n_{\vec{k}\lambda}\rangle = n_{\vec{k}\lambda}|n_{\vec{k}\lambda}\rangle$$

Integer eigenvalues $n_{\vec{k}\lambda}$: the degree of excitation of the mode $(\vec{k},\lambda)$

= "number of photons in this mode"

but there are many modes : each mode is a quantum harm. osc.

General number state

$$|n_{\vec{k}_1 1}\, n_{\vec{k}_1 2}\, n_{\vec{k}_2 1}\, n_{\vec{k}_2 2} \cdots\rangle = |\{n_{\vec{k}\lambda}\}\rangle$$

A completely general state is expressed in this basis as:

$$|\psi\rangle = \sum_{n_{\vec{k}_1 1}=0}^{\infty}\sum_{n_{\vec{k}_1 2}=0}^{\infty}\sum_{n_{\vec{k}_2 1}=0}^{\infty}\cdots\, C_{\{n_{\vec{k}\lambda}\}}\,|\{n_{\vec{k}\lambda}\}\rangle$$

complex amplitudes.

- Each number state is specified by how many photons there are in each mode.

- But there is no conservation law for photons. therefore we allow for a general $|\psi\rangle$ which is not an eigenstate of all the $n_{\vec{k}\lambda}$.

[That's why we cannot put up wavefunction and Schrödinger equation for photons as we do for particles]

Vector potential operator

$$\vec{A}(\vec{r},t) = \sum_{\vec{k}\lambda} \vec{e}_{\vec{k}\lambda} \sqrt{\frac{\hbar}{2\varepsilon_0 V \omega_k}} \left[\hat{a}_{\vec{k}\lambda} e^{-i(\omega_k t - \vec{k}\cdot\vec{r})} + \hat{a}^+_{\vec{k}\lambda} e^{i(\omega_k t - \vec{k}\cdot\vec{r})} \right]$$

the transverse electric field operator

$$\hat{\vec{E}}_T(\vec{r},t) = \hat{\vec{E}}_T^+ + \hat{\vec{E}}_T^- \qquad \{\text{it says "plus", not "dagger"}\}$$

$$\hat{\vec{E}}_T^+ = \sum_{\vec{k}\lambda} \vec{e}_{\vec{k}\lambda} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 \omega}}\, \hat{a}_{\vec{k}\lambda} e^{-i\chi(\vec{r},t)}$$

$$\hat{\vec{E}}_T^- = \sum_{\vec{k}\lambda} \vec{e}_{\vec{k}\lambda} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\, \hat{a}^+_{\vec{k}\lambda} e^{i\chi(\vec{r},t)}$$

(the "positive- & negative-frequency parts")

short-hand for phase:

$$\chi(\vec{r},t) = \omega_k t - \vec{k}\cdot\vec{r} - \frac{\pi}{2} \qquad \left(\frac{\pi}{2} \text{ for the factor } i\right)$$

Similarly

$$\vec{B}(\vec{r},t) = \hat{\vec{B}}^+(\vec{r},t) + \hat{\vec{B}}^-(\vec{r},t)$$

$$\hat{\vec{B}}^+(\vec{r},t) = \sum_{\vec{k}\lambda} \vec{k}\times\vec{e}_{\vec{k}\lambda} \sqrt{\frac{\hbar}{2\varepsilon_0\omega_k V}}\, \hat{a}_{\vec{k}\lambda} e^{-i\chi_k(\vec{r},t)}$$

$$\hat{\vec{B}}^-(\vec{r},t) = \sum_{\vec{k}\lambda} \vec{k}\times\vec{e}_{\vec{k}\lambda} \sqrt{\frac{\hbar}{2\varepsilon_0\omega_k V}}\, \hat{a}^+_{\vec{k}\lambda} e^{i\chi_k(\vec{r},t)}$$

Book Eqs. (4.4.13–4.4.18)

Now define something new:

Quadrature operators

corresponding to $\hat{q}$ and $\hat{p}$ of the h.o.

$$\hat{X}_{\vec{k}\lambda} = \frac{1}{2}\left(\hat{a}^+_{\vec{k}\lambda} + \hat{a}_{\vec{k}\lambda}\right) \qquad \hat{Y}_{\vec{k}\lambda} = \frac{i}{2}\left(\hat{a}^+_{\vec{k}\lambda} - \hat{a}_{\vec{k}\lambda}\right)$$

Hermitian.

$$\left[\hat{X}_{\vec{k}\lambda}, \hat{Y}_{\vec{k}\lambda}\right] = \frac{i}{4}\left[(\hat{a}^+ + \hat{a}), (\hat{a}^+ - \hat{a})\right]$$

$$= \frac{i}{4}\left([\hat{a},\hat{a}^+] - [\hat{a}^+,\hat{a}]\right) = \boxed{\frac{i}{2}}$$

Note that we can write

$$\hat{\vec{E}}_T(\vec{r},t) = \sum_{\vec{k},\lambda} \vec{e}_{\vec{k}\lambda} \sqrt{\frac{2\hbar\omega_k}{\varepsilon_0 V}}\left(\hat{X}_{\vec{k}\lambda}\cos\chi_{\vec{k}}(\vec{r},t) + \hat{Y}_{\vec{k}\lambda}\sin\chi_{\vec{k}}(\vec{r},t)\right)$$

a more natural representation for some purposes.

the hamiltonian

$$\hat{H}_R = \sum_{\vec{k}\lambda} \hbar\omega_k\left(\hat{X}^2_{\vec{k}\lambda} + \hat{Y}^2_{\vec{k}\lambda}\right)$$

Why only transverse $\vec{E}$-field?

~ The longitudinal can also be quantized best done in relativistic theory.

Produces in fact only small corrections to the physics we are interested in.

See any textbook on quantum field theory, such as Mandl & Shaw.

Vacuum state.

Recall $\hat{H}_R = \sum_{\vec{k}\lambda} \hbar\omega_k \left(\hat{a}^+_{\vec{k}\lambda}\hat{a}_{\vec{k}\lambda} + \frac{1}{2}\right)$

For a number state we have

$$\hat{H}_R \,|\{n_{\vec{k}\lambda}\}\rangle = \underbrace{\sum_{\vec{k}\lambda} \hbar\omega_k \left(n_{\vec{k}\lambda} + \tfrac{1}{2}\right)}_{\text{the eigenenergy}} |\{n_{\vec{k}\lambda}\}\rangle$$

Ground state: No photons in any mode,

$n_{\vec{k}\lambda} = 0$ all $\vec{k}\lambda$

$\hat{a}_{\vec{k}\lambda}\,|\{0\}\rangle = 0$ all $\vec{k}\lambda$, "Vacuum state"

Energy of the vacuum state is

$E_0 = \sum_{\vec{k}\lambda} \frac{1}{2}\hbar\omega_k =$ infinite!

[More about this in chap. 6: still measurable]

We can define

$$\hat{H}_R\,|\{n_{\vec{k}\lambda}\}\rangle = E\,|\{n_{\vec{k}\lambda}\}\rangle = (E_R + E_0)\,|\{n_{\vec{k}\lambda}\}\rangle$$

where $E_R = \sum_{\vec{k}\lambda} n_{\vec{k}\lambda}\hbar\omega_k$

is finite if the number of photons is finite.

We will see that $E_R \Leftrightarrow$ irradiance.

(4.8) Atom-radiation interaction

- Dipole approximation – derivation.

Atom: Charge density

$$\sigma(\vec{r}) = \underbrace{-e\sum_{\alpha=1}^{Z}\delta(\vec{r}-\vec{r}_\alpha)}_{\text{electrons}} + \underbrace{Ze\delta(\vec{r})}_{\text{nucleus at } \vec{r}=0}$$

the polarization obeys the equation

$$-\nabla\cdot\vec{P}(\vec{r}) = \sigma(\vec{r})$$

Solution

$$\vec{P}(\vec{r}) = -e\sum_\alpha \vec{r}_\alpha \int_0^1 d\zeta\, \delta(\vec{r}-\zeta\vec{r}_\alpha)$$

The potential energy is

$$V_E = -\int d\vec{r}\, \vec{P}(\vec{r})\cdot\vec{E}_T(\vec{r}) \qquad \left\{\begin{matrix}\text{No contribution} \\ \text{from } \vec{E}_L.\end{matrix}\right\}$$

$$= e\sum_\alpha \int_0^1 d\zeta\, \vec{r}_\alpha\cdot\vec{E}_T(\zeta\vec{r}_\alpha)$$

Taylor expand in ζ:

$$V_E = e\sum_\alpha \int_0^1 d\zeta\, \vec{r}_\alpha\cdot\vec{E}_T(0) + (\zeta\vec{r}_\alpha\cdot\nabla)(\vec{r}_\alpha\cdot\vec{E}_T(0)) + \frac{1}{2!}(\ldots)\ldots$$

$$= \underbrace{e\sum_\alpha \vec{r}_\alpha\cdot\vec{E}_T(0)}_{\text{Dipole term}} + \underbrace{\frac{1}{2}(\vec{r}_\alpha\cdot\nabla)(\vec{r}_\alpha\cdot\vec{E}_T(0))}_{\text{Quadrupole term}} + \ldots$$

Order-of-magnitude estimates:

$|\vec{r}_\alpha| \sim a_B$ Bohr radius. $a_B = \dfrac{\hbar^2 \varepsilon_0}{\pi m_e e^2}$

$\nabla E \sim \dfrac{E}{\lambda}$ where $\lambda = \dfrac{2\pi c}{\omega}$

and typically we deal with ω corresp. to energy level differences in an atom
which are all the order of Hydrogen ground state

$$\hbar\omega \sim \frac{e^2}{8\pi\varepsilon_0 a_B}$$

so $\dfrac{\text{quadrupole} \to E_Q}{\text{dipole} \to E_D} \sim \dfrac{2e a_B^2 E/\lambda}{e a_B E} \sim \dfrac{2a_B}{\lambda} \sim \dfrac{2e^2 a_B}{c\hbar\, 8\pi\varepsilon_0 a_B}$

$$\boxed{\sim \frac{e^2}{4\pi\varepsilon_0 \hbar c} = \alpha \approx \frac{1}{137}}$$

the fine-structure constant.

So typically, $E_D \gg E_Q$

In the same manner, find that the magnetic energy to lowest order is α times smaller than E_D.

$\Rightarrow$ work with dipole approx $H = H_{ED}$.

The complete Hamiltonian is now

$$\hat{H} = \hat{H}_A + \hat{H}_R + \hat{H}_{ED}$$

$\hat{H}_A$ = as before. Coulomb between electrons & nucleus

$\hat{H}_R$ = we just derived it: $\sum \hbar\omega_k (\hat{n}_{k\lambda} + \frac{1}{2})$

$$\hat{H}_{ED} = e \sum_{\alpha=1}^{Z} \vec{r}_\alpha \cdot \hat{\vec{E}}_T(0) = e \hat{\vec{D}} \cdot \hat{\vec{E}}_T(0)$$

(London does it in a more roundabout way:

Writes $\hat{H} = \sum_\alpha \frac{1}{2m} (\hat{\vec{p}}_\alpha + e \hat{\vec{A}}(\hat{\vec{r}}_\alpha))^2$

then do some algebra & retain the dipole term.)