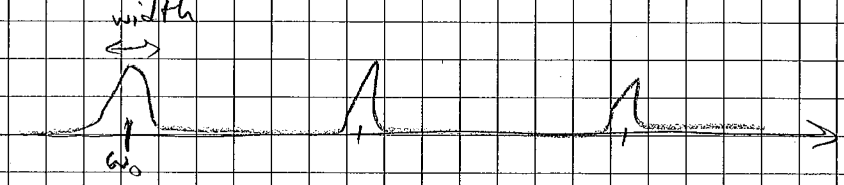
2.5 - 2.7

4 Broadening & Bloch equations

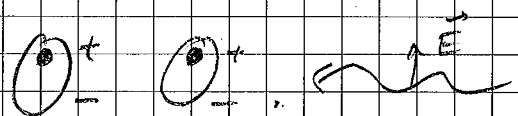
Atoms absorb & emit light
even if it is not resonant
- if we take A_{21} into account!
$\Rightarrow$ linewidth

Spectrum



How? :

An EM-wave will polarize the atom



Def. Polarization $\vec{P}(t) = \frac{N}{V}\left(-e\vec{D}(t)\right)$

Assume $\vec{E} \parallel \vec{e}_x$, then the magnitude

$$P(t) = -e\frac{N}{V}X(t) \qquad [\text{Recall } X = \text{c.m.}]$$

Write
$$\boxed{P(t) = \frac{1}{2}\varepsilon_0 E_0 \left\{\chi(\omega)e^{i\omega t} + \chi(-\omega)e^{-i\omega t}\right\}}$$

where we defined

susceptibility

$$\boxed{\chi(\omega) = \underbrace{\chi'(\omega)}_{=\mathrm{Re}\,\chi} + i\underbrace{\chi''(\omega)}_{=\mathrm{Im}\,\chi}}$$

χ' real part $\leftrightarrow$ refraction

χ'' im part $\leftrightarrow$ absorption

Solve Schrödinger eq.

Include spontaneous emission by hand:

$$i\frac{\partial C_2}{\partial t} = \Omega \cos\omega t \, e^{i\omega_0 t} C_1 - i\gamma_{sp} C_2$$

where $\gamma_{sp} = \frac{A_{21}}{2} = \frac{1}{2} \cdot \frac{e^2\omega_0^3 D_{12}^2}{3\pi\varepsilon_0 \hbar c^3}$

spontaneous decay rate

natural linewidth

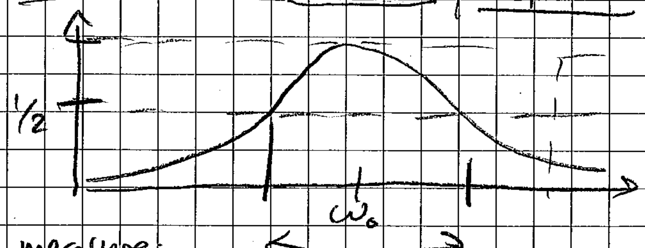
Obtain

$$C_2(t) = -\frac{1}{2}\Omega^* \left\{ \underbrace{\frac{e^{i(\omega_0+\omega)t}}{\omega_0+\omega-i\gamma_{sp}}}_{\to 0 \text{ RWA}} + \frac{e^{i(\omega_0-\omega)t}}{\omega_0-\omega-i\gamma_{sp}} \right\}$$

Finally obtain

$$\boxed{\chi''(\omega) = \frac{2\pi N c^3}{\omega_0^3 V} \frac{\gamma_{sp}^2}{(\omega_0-\omega)^2+\gamma_{sp}^2}}$$

a Lorentzian line profile.



Important measure:

Full width at half maximum

FWHM $= 2\gamma_{sp}$

4-B Qualitative (& sloppy!) explanation:

Energy-time uncertainty relation

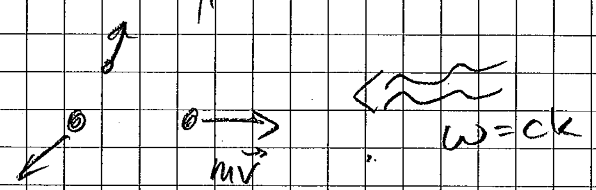
$$\Delta E \Delta t \geq \hbar$$

Δt here: $\tau_R = \frac{1}{\gamma_{sp}}$ lifetime of excited state

ΔE: linewidth ~ "uncertainty in energy of exc state"

Doppler broadening

Atoms have different velocities



Light is Doppler shifted:

$$\omega_0 \longrightarrow \omega = \omega_0 - v_1 k$$

Atoms are Maxwell-Boltzmann distributed:

$$P(v_1) \propto e^{-\frac{Mv_1^2/2}{k_B T}}$$

$\Rightarrow$ absorbed light is distributed according to

$$P(\omega) \propto e^{-\frac{M}{2k_BT}\left(c^2 \frac{(\omega-\omega_0)^2}{\omega_0^2}\right)}$$

a Gaussian distribution.

Note Fig. 2.4 Difference

Typically:

Homogeneous broadening - Same for all atoms
typically Lorentzian

Inhomogeneous broadening - different atoms have
different resonances
Typically Gaussian

There is also collisional broadening
Power — " —

To obtain the true lineshape,
combine the effects by convolution

$$F(\omega) = \int_{-\infty}^{\infty} d\nu \; F_1(\nu) F_2(\omega+\omega_0-\nu)$$

4. The Optical Bloch equations

2-7
2-9 A more convenient formulation for some QM processes is the <u>density matrix</u>.

Can be adapted to - many-body
statistical problems
damping

For two states, define

$$\left.\begin{aligned}\rho_{11}(t) &= |C_1(t)|^2 \\ \rho_{22}(t) &= |C_2(t)|^2\end{aligned}\right\} \text{populations}$$

$$\left.\begin{aligned}\rho_{12}(t) &= C_1(t)^* C_2(t) \\ \rho_{21}(t) &= \rho_{12}(t)^*\end{aligned}\right\} \text{coherences}$$

or, if you will,

$$\rho = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}\begin{pmatrix} C_1^* & C_2^* \end{pmatrix} \quad \text{direct product}$$

$$= \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$$

<u>Note</u>: 1) $|\rho_{12}|^2 = |\rho_{21}|^2 = \rho_{12}\rho_{21} = |C_1|^2|C_2|^2 = \rho_{11}\rho_{22}$

2) $\rho_{11} + \rho_{22} = 1$ normalization

for ordinary QM, "pure states"

4-6 Observe puzzle:

Schrödinger picture

Degrees of freedom		Constraints
Re C_1 or	$\lvert C_1\rvert$	$\lvert C_1\rvert^2 + \lvert C_2\rvert^2 = 1$
Im C_1	arg C_1	
Re C_2	$\lvert C_2\rvert$	
Im C_2	arg C_2	

$$4 \quad - \quad 1 \quad = 3$$

Density matrix

Degrees of fr.	Constraints
ρ_{11}	$\lvert\rho_{12}\rvert^2 = \rho_{11}\rho_{22}$
ρ_{22}	$\rho_{11} + \rho_{22} = 1$
$\left.\begin{matrix}\text{Re } \rho_{12} \\ \text{Im } \rho_{12}\end{matrix}\right\}$ or $\left\{\begin{matrix}\lvert\rho_{12}\rvert \\ \arg \rho_{12}\end{matrix}\right.$	

$$4 \quad - \quad 2 \quad = \quad 2$$

Does that mean that we have lost information in going to density matrix?

Answer: No. Overall phase of wavefunction is redundant

$$|\psi\rangle = C_1|1\rangle + C_2|2\rangle \rightarrow e^{i\alpha}(C_1|1\rangle + C_2|2\rangle)$$

Only relative phase between C_1 and C_2 matters

This is encoded in density matrix.

4-7 Derive eqs of motion for ρ

Use

$$i\frac{dC_1}{dt} = \Omega\cos(\omega t)e^{-i\omega_0 t}C_2$$

$$i\frac{dC_2}{dt} = \Omega\cos(\omega t)e^{i\omega_0 t}C_1$$

and chain rule

$$\frac{d\rho_{11}}{dt} = \frac{d}{dt}\left(C_1^* C_1\right) = C_1^*\frac{dC_1}{dt} + C_1\frac{dC_1^*}{dt}$$

$$= C_1^*\frac{1}{i}\Omega\cos(\omega t)e^{-i\omega_0 t}C_2 + C_1\frac{1}{i^*}\Omega\cos(\omega t)e^{i\omega_0 t}C_2^*$$

$$= -i\Omega\cos(\omega t)e^{-i\omega_0 t}\rho_{21} + i\Omega\cos(\omega t)e^{i\omega_0 t}\rho_{12}$$

$$\boxed{\frac{d\rho_{11}}{dt} = i\Omega\cos(\omega t)\left(e^{i\omega_0 t}\rho_{12} - e^{-i\omega_0 t}\rho_{21}\right)}$$

Use $\rho_{22} = 1 - \rho_{11}$

$$\Rightarrow \boxed{\frac{d\rho_{22}}{dt} = -\frac{d\rho_{11}}{dt}}$$

$$\frac{d\rho_{12}}{dt} = C_2^*\frac{1}{i}\Omega\cos(\omega t)e^{-i\omega_0 t}C_2 + C_1\frac{1}{i^*}\Omega\cos(\omega t)e^{-i\omega_0 t}C_1^*$$

$$\boxed{\frac{d\rho_{12}}{dt} = i\Omega\cos(\omega t)e^{-i\omega_0 t}\left(\rho_{11} - \rho_{22}\right)}$$

$$\boxed{\frac{d\rho_{21}}{dt} = \left(\frac{d\rho_{12}}{dt}\right)^*}$$

One more step ... $\rightarrow$

$$\frac{d\rho_{11}}{dt} = i\Omega \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)\left(e^{i\omega_0 t}\rho_{12} - e^{-i\omega_0 t}\rho_{21}\right)$$

<u>RWA</u>: Discard fast oscillating terms
- because we have seen they give <u>small</u> contrib.

discard $e^{i(\omega+\omega_0)t}$ and $e^{-i(\omega+\omega_0)t}$

$$\left\{\begin{aligned}
\frac{d\rho_{11}}{dt} &= \frac{i\Omega}{2}\left(e^{i(\omega_0-\omega)t}\rho_{12} - e^{-i(\omega_0-\omega)t}\rho_{21}\right) \\
\frac{d\rho_{22}}{dt} &= -\frac{d\rho_{11}}{dt} \\
\frac{d\rho_{12}}{dt} &= \frac{i\Omega}{2} e^{-i(\omega_0-\omega)t}\left(\rho_{11} - \rho_{22}\right) \\
\frac{d\rho_{21}}{dt} &= \left(\frac{d\rho_{12}}{dt}\right)^*
\end{aligned}\right.$$

<u>The O.B.E.</u>

<u>One more simplification</u>

Define $\tilde{\rho}_{12} = e^{i(\omega_0-\omega)t}\rho_{12}$ $\quad \tilde{\rho}_{11} = \rho_{11}$

$\tilde{\rho}_{21} = e^{-i(\omega_0-\omega)t}\rho_{21}$ $\quad \tilde{\rho}_{22} = \rho_{22}$

just so notation gets prettier

Insert

$$\boxed{\frac{d\tilde{\rho}_{11}}{dt} = \frac{i\Omega}{2}\left(\tilde{\rho}_{12} - \tilde{\rho}_{21}\right)}$$ by inspection

$$\frac{d\tilde{\rho}_{12}}{dt} = \frac{d}{dt}\left(\rho_{12} e^{i(\omega_0-\omega)t}\right) =$$

$$= i(\omega_0-\omega)\rho_{12} e^{i(\omega_0-\omega)t} + e^{i(\omega_0-\omega)t}\frac{d\rho_{12}}{dt}$$

$$\boxed{\frac{d\tilde{\rho}_{12}}{dt} = i(\omega_0-\omega)\tilde{\rho}_{12} + \frac{i\Omega}{2}\left(\tilde{\rho}_{11}-\tilde{\rho}_{22}\right)}$$

See if we can find a solution of the form

$$\tilde{\rho}_{ij}(t) = \tilde{\rho}_{ij}(0)e^{\lambda t}$$

$$\frac{i\Omega}{2}\left(\tilde{\rho}_{12}(0)-\tilde{\rho}_{21}(0)\right) = \lambda\tilde{\rho}_{11}(0)$$

$$-\frac{i\Omega}{2}\left(\tilde{\rho}_{12}(0)-\tilde{\rho}_{21}(0)\right) = \lambda\tilde{\rho}_{22}(0)$$

$$i(\omega_0-\omega)\tilde{\rho}_{12}(0) + \frac{i\Omega}{2}\left(\tilde{\rho}_{11}(0)-\tilde{\rho}_{22}(0)\right) = \lambda\tilde{\rho}_{12}(0)$$

$$-i(\omega_0-\omega)\tilde{\rho}_{21}(0) - \frac{i\Omega}{2}\left(\tilde{\rho}_{11}(0)-\tilde{\rho}_{22}(0)\right) = \lambda\tilde{\rho}_{21}(0)$$

an <u>eigenvalue problem</u>!

$$\begin{pmatrix} -\lambda & 0 & \frac{i\Omega}{2} & -\frac{i\Omega}{2} \\ 0 & -\lambda & -\frac{i\Omega}{2} & \frac{i\Omega}{2} \\ \frac{i\Omega}{2} & -\frac{i\Omega}{2} & i(\omega_0-\omega)-\lambda & 0 \\ -\frac{i\Omega}{2} & \frac{i\Omega}{2} & 0 & -i(\omega_0-\omega)-\lambda \end{pmatrix}\begin{pmatrix} \tilde{\rho}_{11}(0) \\ \tilde{\rho}_{22}(0) \\ \tilde{\rho}_{12}(0) \\ \tilde{\rho}_{21}(0) \end{pmatrix} = 0$$

4-10 Compute the 4x4 determinant

$$-\lambda \begin{vmatrix} -\lambda & -\frac{i\Omega}{2} & \frac{i\Omega}{2} \\ -\frac{i\Omega}{2} & i(\omega_0-\omega)-\lambda & 0 \\ \frac{i\Omega}{2} & 0 & -i(\omega_0-\omega)-\lambda \end{vmatrix} + \frac{1}{2}i\Omega \begin{vmatrix} 0 & -\lambda & \frac{i\Omega}{2} \\ \frac{i\Omega}{2} & -\frac{i\Omega}{2} & 0 \\ -\frac{i\Omega}{2} & \frac{i\Omega}{2} & -i(\omega_0-\omega)-\lambda \end{vmatrix}$$

$$+\frac{1}{2}i\Omega \begin{vmatrix} 0 & -\lambda & -\frac{i\Omega}{2} \\ \frac{i\Omega}{2} & -\frac{i\Omega}{2} & i(\omega_0-\omega)-\lambda \\ -\frac{i\Omega}{2} & \frac{i\Omega}{2} & 0 \end{vmatrix} = 0$$

$$= -\lambda \left\{ -\lambda \left(i(\omega_0-\omega)-\lambda\right)\left(-i(\omega_0-\omega)-\lambda\right) - \left(\frac{i\Omega}{2}\right)^2 \left(i(\omega_0-\omega)-\lambda\right) - \left(\frac{i\Omega}{2}\right)^2 \left(-i(\omega_0-\omega)-\lambda\right) \right\}$$

$$+\frac{1}{2}i\Omega \left\{ \left(\frac{i\Omega}{2}\right)^3 - (-\lambda)\frac{i\Omega}{2}\left(-i(\omega_0-\omega)-\lambda\right) - \left(\frac{i\Omega}{2}\right)^3 \right\}$$

$$+\frac{1}{2}i\Omega \left\{ -\left(\frac{i\Omega}{2}\right)^3 - \frac{i\Omega}{2}(-\lambda)\left(i(\omega_0-\omega)-\lambda\right) - (-1)\left(\frac{i\Omega}{2}\right)^3 \right\}$$

$$= \lambda^2\left(\lambda^2 - \left(i(\omega_0-\omega)\right)^2\right) - \lambda^2 \cdot 2\left(\frac{i\Omega}{2}\right)^2$$

$$+ 2\left(\frac{i\Omega}{2}\right)^2 \lambda^2 \cdot (-1)$$

$$= \lambda^2 \left\{ \lambda^2 + (\omega_0-\omega)^2 + 4\frac{\Omega^2}{4} \right\}$$

$$= \lambda^2 \left\{ \lambda^2 + (\omega_0-\omega)^2 + \Omega^2 \right\} = 0$$

Solutions: $\begin{cases} \lambda = 0 \\ \lambda = \pm i\sqrt{(\omega_0-\omega)^2+\Omega^2} = \pm i\Omega_{gen} \end{cases}$

"the generalized Rabi frequency"

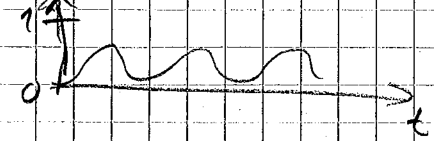
Notation

General solution

$$\rho_{ij}(t) = \rho_{ij}^{(1)} + \rho_{ij}^{(2)} e^{i\Omega_{gen}t} + \rho_{ij}^{(3)} e^{-i\Omega_{gen}t}$$

Specific solution for $\rho_{22}(0) = 0$, $\rho_{22}'(0) = 0$:

$$\rho_{22}(t) = \left(\frac{\Omega}{\Omega_{gen}}\right)^2 \sin^2\left(\tfrac{1}{2}\Omega_{gen} t\right)$$



$$= \frac{1}{2}\left(\frac{\Omega}{\Omega_{gen}}\right)^2 \left(1 - \frac{1}{2} e^{i\Omega_{gen}t} - \frac{1}{2} e^{-i\Omega_{gen}t}\right)$$

$$\left(\rho_{22}^{(1)} = \frac{1}{2}\left(\frac{\Omega}{\Omega_{gen}}\right)^2 \quad \rho_{22}^{(2)} = -\frac{1}{4}\left(\frac{\Omega}{\Omega_{gen}}\right)^2 \quad \rho_{22}^{(3)} = -\frac{1}{4}\left(\frac{\Omega}{\Omega_{gen}}\right)^2\right)$$

<u>Note</u>: At resonance: $\Omega_{gen} = \sqrt{(\omega_0 - \omega)^2 + \Omega^2} = \Omega$

then
$$\boxed{\begin{aligned} \rho_{22}(t) &= \sin^2\left(\tfrac{1}{2}\Omega t\right) \\ \rho_{11}(t) &= \cos^2\left(\tfrac{1}{2}\Omega t\right) \end{aligned}}$$

<u>"Rabi oscillations"</u>, "optical nutation"

with the Rabi frequency $\Omega = \frac{1}{\hbar} \vec{E} \cdot \vec{D}_{12}$

<u>Ex</u>: Let the light stay on for a time T

s.t. $\Omega T = \pi$

then $\rho_{22}(T) = 1$

$\rho_{11}(T) = 0$

whole population is transferred.

A "π-pulse".

<u>Notation</u>:

London	E.L. & rest of world
2ν	Ω
Ω	Ω_{gen}

4-12 Ex. 2. Let $\Omega T = \frac{\pi}{2}$

Then $\rho_{22}(T) = \frac{1}{2}$

$\rho_{11}(T) = \frac{1}{2}$

the atoms are put in a coherent superposition of $|1\rangle$ and $|2\rangle$

"$\frac{\pi}{2}$-pulse"

useful tool for atomic physics experiments.

Spontaneous emission

Again, in the semiclassical picture have to introduce spont. em "by hand"

$$\frac{d\tilde{\rho}_{22}}{dt} = -\frac{i\Omega}{2}\left(\tilde{\rho}_{12} - \tilde{\rho}_{21}\right) - 2\gamma_{sp}\tilde{\rho}_{22} = -\frac{d\tilde{\rho}_{11}}{dt}$$

$$\frac{d\tilde{\rho}_{12}}{dt} = \frac{i\Omega}{2}\left(\tilde{\rho}_{11} - \tilde{\rho}_{22}\right) + \left[i(\omega_0 - \omega) - \gamma_{sp}\right]\tilde{\rho}_{12} = \left(\frac{d\tilde{\rho}_{21}}{dt}\right)^*$$