# 14. Laser fluctuations

What are the probabilities $P(n)$ for the laser

Steady state: $N_2 T_{st} P(n-1) = T_{cav} P(n)$  3-level

$$\Rightarrow \dots \Rightarrow \quad P(n) = \frac{(C n_s)^n (n_s-1)!}{(n_s+n-1)!} P(0)$$

Simplifies in two limits

* Below threshold: $C < 1$

  only $n \ll n_s$ are populated

  then $(n_s+n-1)! \approx n_s^n (n_s-1)!$

  $\Rightarrow \quad P(n) = C^n P(0) = C^n(1-C)$ geometric distribution

  again $\langle n \rangle = \frac{C}{1-C}$

  and $g^{(2)}(0) = 2$

  $\rightarrow$ Chaotic light.

  The mode is dominated by spontaneous emission.

* Above threshold, $C > 1$

  $n \gg 1$ so $n-1 \approx n$

$$P(n) = \frac{(C n_s)^n n_s!}{(n_s+n)!} P(0)$$

$$\langle n \rangle = \sum_{n=0}^{\infty} n P(n) = \sum_n (n_s + n - n_s) \frac{(C n_s)^n n_s!}{(n_s+n)!} P(0)$$

$$= \sum_n \frac{(C n_s)^n n_s!}{(n_s+n-1)!} P(0) - n_s \sum_n \frac{(C n_s)^n n_s!}{(n_s+n)!} P(0)$$

$$\langle n \rangle = (G n_s) \underbrace{\sum_n P(n-1)}_{=1} - n_s \underbrace{\sum_n P(n)}_{=1}$$

$$\langle n \rangle = (G-1)\, n_s$$

then

$$P(n) = \frac{(n_s + \langle n \rangle)^n\, n_s!}{(n_s+n)!} P(0)$$

$$\boxed{P(n) \approx \frac{1}{\sqrt{2\pi (n_s + \langle n \rangle)}}\, e^{-(n-\langle n \rangle)^2 / 2(n_s + \langle n \rangle)}}$$

Variance: $(\Delta n)^2 = n_s + \langle n \rangle$

Compare coherent state

$$P(n) = \frac{1}{\sqrt{2\pi \langle n \rangle}}\, e^{-\frac{(n-\langle n \rangle)^2}{2\langle n \rangle}} \quad \text{coherent}$$

This state is super-Poissonian

but close to Poissonian if $\langle n \rangle \gg n_s$

and $g^{(2)}(0) = \frac{n_s}{\langle n \rangle^2} + 1 \approx 1$

$n_s$ is a little extra amplitude noise due to spont. emission.

Rate of spontaneous emission into this mode

is $N_2 \Gamma_{st}' = \left(\begin{array}{c}\text{\# atoms that}\\ \text{can decay}\end{array}\right) \cdot \left(\begin{array}{c}\text{decay rate of}\\ \text{each atom}\end{array}\right)$

But we found that above threshold,

$$N_2 \Gamma_{st} = \Gamma_{cav}$$

⇒ Spontaneous photons are created at the same rate as they are lost from the cavity

→ on average, there is <u>one</u> noise photon in the lasing mode.

*However, the <u>phase</u> will diffuse over long times

# Atom-radiation dynamics (§7.7-7.9)

Motivation: Have done rate equations.
Necessary for large-$N$ problems - statistical method.
Now: try full quantum evolution
Suitable for problems with one atom

Work in Heisenberg picture. (4.9.24)

$$\hat{H} = \hbar\omega_0 \hat{\pi}^\dagger(t)\hat{\pi}(t) \qquad \text{(atom)}$$
$$+ \sum_{\vec{k}\lambda} \hbar\omega_k \left(\hat{a}^\dagger_{\vec{k}\lambda}(t)\hat{a}_{\vec{k}\lambda}(t) + \frac{1}{2}\right) \qquad \text{(radiation)}$$
$$+ i \sum_{\vec{k}\lambda} \hbar g_{\vec{k}\lambda} \left\{ \hat{\pi}^\dagger(t)\hat{a}_{\vec{k}\lambda}(t) e^{i\vec{k}\cdot\vec{R}} - \hat{a}^\dagger_{\vec{k}\lambda}(t)\hat{\pi}(t) e^{-i\vec{k}\cdot\vec{R}} \right\}$$
(electric dipole)

Put up equations of motion for the operators

$$i\frac{d\hat{a}_{\vec{k}\lambda}}{dt} = \frac{1}{\hbar}\left[\hat{a}_{\vec{k}\lambda}, \hat{H}\right] = \omega_k \hat{a}_{\vec{k}\lambda}(t) - i g_{\vec{k}\lambda}\hat{\pi}(t) e^{-i\vec{k}\cdot\vec{R}}$$

Formal solution:

$$\hat{a}_{\vec{k}\lambda}(t) = e^{-i\omega_k t}\left[\hat{a}_{\vec{k}\lambda}(0) - g_{\vec{k}\lambda}\int_0^t dt' \hat{\pi}(t') e^{-i\vec{k}\cdot\vec{R} + i\omega_k t'}\right] \qquad (1)$$

$$i\frac{d\hat{\pi}}{dt} = \omega_0\hat{\pi}(t) - i\sum_{\vec{k}\lambda} g_{\vec{k}\lambda}\left[2\hat{\pi}^\dagger(t)\hat{\pi}(t) - 1\right]\hat{a}_{\vec{k}\lambda}(t) e^{i\vec{k}\cdot\vec{R}}$$

$$\Rightarrow \hat{\pi}(t) = e^{-i\omega_0 t}\Big[\underbrace{\hat{\pi}(0)}_{\text{free atom}} - \underbrace{\sum_{\vec{k}\lambda} g_{\vec{k}\lambda}\int_0^t dt' \left(2\hat{\pi}^\dagger(t')\hat{\pi}(t') - 1\right)\hat{a}_{\vec{k}\lambda}(t') \times e^{i\vec{k}\cdot\vec{R} + i\omega_0 t'}}_{\text{coupling to field}}\Big]$$

How do we obtain a solution on closed form?
- Proceed by iteration.
Assume $g_{k\lambda}$ small.

Example: Calculate the time dependence of
$$\langle \hat{\pi}^+ \hat{\pi} \rangle = \langle \psi | 2 \rangle \langle 2 | \psi \rangle = |\langle 2 | \psi \rangle|^2$$
= degree of excitation.

$$\frac{d}{dt}\left(\hat{\pi}^+(t)\hat{\pi}(t)\right) = \sum_{k\lambda} g_{k\lambda}\left[\hat{\pi}^+(t)\hat{a}_{k\lambda}(t)e^{i\vec{k}\cdot\vec{R}} + \hat{a}^+_{k\lambda}\hat{\pi}(t)\, e^{-i\vec{k}\cdot\vec{R}}\right] \qquad (3)$$

[again, this can be found by working out the commutator]

Insert $\hat{\pi}^+, \hat{\pi}, \hat{a}^+, \hat{a}$ to lowest order in $g_{k\lambda}$

$$(2) \Rightarrow \hat{\pi}(t) = e^{-i\omega_0 t}\hat{\pi}(0) \Rightarrow \hat{\pi}(t') = e^{-i\omega_0(t'-t)}\hat{\pi}(t)$$

Into (1)

$$(1) \Rightarrow \hat{a}_{\vec{k}\lambda}(t) = e^{-i\omega_k t}\left(\hat{a}_{k\lambda}(0) - g_{k\lambda}\hat{\pi}(t)e^{i\omega_0 t}e^{-i\vec{k}\cdot\vec{R}}\int_0^t dt'\, e^{i(\omega_k-\omega_0)t'}\right.$$

$$(3) \Rightarrow \frac{d}{dt}\left(\hat{\pi}^+\hat{\pi}\right) =$$
$$\hat{\pi}^+(t)\sum_{\vec{k}\lambda} g_{\vec{k}\lambda}\left[\hat{a}_{k\lambda}(0)e^{i\vec{k}\cdot\vec{R}-i\omega_k t} - g_{\vec{k}\lambda}\hat{\pi}(t)\int_0^t dt'\, e^{i(\omega_k-\omega_0)(t'-t)}\right] + \text{h.c.}$$

Now take the expectation value in vacuum for the radiation:
$|0_{rad}\rangle$, (no photons)
$\langle 0_{rad}|\hat{a}_{k\lambda}(t)|0_{rad}\rangle = 0$, so

$$\frac{d}{dt}\langle 0_{rad}|\hat{\pi}^+\hat{\pi}|0_{rad}\rangle = -\langle 0_{rad}|\hat{\pi}^+(t)\hat{\pi}(t)|0_{rad}\rangle \sum_{k\lambda} g^2_{k\lambda}\cdot 2\underbrace{\frac{\sin[(\omega_k-\omega_0)t]}{\omega_k-\omega_0}}_{\to \pi\delta(\omega_k-\omega_0)\text{ as } t\to\infty}$$

$$\therefore \frac{d}{dt}\underbrace{\langle 0_r|\hat{\pi}^{+}\hat{\pi}|0_r\rangle}_{=\rho_{22}} = -\langle 0_r|\hat{\pi}^{+}(t)\hat{\pi}(t)|0_r\rangle \cdot \underbrace{2\pi\sum_{k\lambda} g_{k\lambda}^2\,\delta(\omega_k-\omega_0)}_{=2\gamma_{sp}}$$

$\therefore$ We have reproduced the result

$$\frac{d\rho_{22}}{dt} = -2\gamma_{sp}\rho_{22} \quad \text{in vacuum.}$$

- What about the rate of atomic transition?

$$\frac{d}{dt}\langle 0_r|\hat{\pi}(t)|0_r\rangle = -\langle 0_r|\hat{\pi}(t)|0_r\rangle \underbrace{\left\{ i\omega_0 + \sum_{k\lambda} g_{k\lambda}^2 \int_0^t dt' e^{i(\omega_k-\omega_0)(t'-t)} \right\}}$$

$$i\omega_0 + \sum_{k\lambda} g_{k\lambda}^2 \left( \frac{1-\cos[(\omega_k-\omega_0)t]}{i(\omega_k-\omega_0)} + \frac{\sin(\omega_k-\omega_0)t}{\omega_k-\omega_0} \right)$$

$$= i\omega_0 - i\underbrace{\left[\sum_{\lambda}\frac{V}{(2\pi)^3}\int d\vec{k}\,\lim_{t\to\infty}\left(\frac{1-\cos[(\omega_k-\omega_0)t]}{i(\omega_k-\omega_0)}\right)\cdot g_{\vec{k}\lambda}\right]}_{\substack{\text{A vacuum correction to the}\\ \text{resonance frequency}\\ \text{"the Lamb shift"}}} + \gamma_{sp}$$

Triumph of QED: this small shift $\sim 10^{-6}$ measured by Lamb in 1950s

★ What is $\hat{\vec{E}}_T^+(\vec r,t)$ in the same approx.?

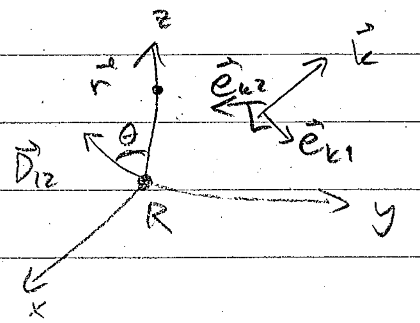
$$\hat{\vec{E}}_T^+(\vec r,t) = i\sum_{\vec k\lambda}\vec e_{k\lambda}\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\,\hat a_{k\lambda}(t)\,e^{i\vec k\cdot\vec r}$$

Insert our exact expression for $\hat a_{k\lambda}(t)$

neglect the $\hat a(0)$ term

$$\hat{\vec{E}}_T^+(\vec r,t) = -i\sum_{\vec k\lambda}\vec e_{k\lambda}\underbrace{\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\,g_{k\lambda}}_{\frac{e}{16\pi^3\varepsilon_0}\omega_k\,\vec e_{k\lambda}\cdot\vec D_{12}}\;e^{i\vec k\cdot(\vec r-\vec R)}\int_0^t dt'\,\hat\pi(t')\,e^{i\omega_k t'}$$

Note:




$\vec r$ is the point of observation

$\vec R$ is the location of the nucleus

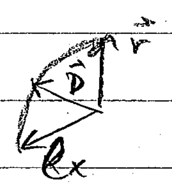
$\vec k \perp \vec e_{k1} \perp \vec e_{k2}$

this is called the source-field expression $E_{SF}$ because it retains only the part coming from the source (=atom), containing $\hat\pi$ not the free-field part $\hat a(0)$

Some more manipulation yields

$$\hat{\vec{E}}_{sf}(\vec r,t) = -\frac{e\,\omega_0^2\,D_{12}\sin\theta}{4\pi\varepsilon_0 c^2\,|r-R|}\,\vec e_x\,\hat\pi\!\left(t-\frac{|\vec r-\vec R|}{c}\right)$$

where $\vec e_x$ is a unit vector for the projection of $\vec D_{12}$ perpendicular to $\vec r-\vec R$.



# Emission by a single driven atom

Assume the light is polarized onto $\vec{e}_x$:

$$\hat{\vec{E}}_{sf} \to \hat{E}_{sf} \text{ scalar.}$$

Coherences

$$g^{(1)}(\tau) = \frac{\langle E^-_{sf}(\vec{r},t)\, E^+_{sf}(\vec{r},t+\tau)\rangle}{\langle E^-_{sf}(r,t) E^+_{sf}(r,t)\rangle} = \frac{\langle \hat{\pi}^+(t)\hat{\pi}(t+\tau)\rangle}{\langle \hat{\pi}^+(t)\hat{\pi}(t)\rangle}$$

$$g^{(2)}(\tau) = \frac{\langle \hat{\pi}^+(t)\, \hat{\pi}^+(t+\tau)\hat{\pi}(t+\tau)\, \hat{\pi}(t)\rangle}{\langle \hat{\pi}^+(t)\hat{\pi}(t)\rangle^2}$$

For zero delay $\tau = 0$

$$g^{(1)}(0) = 1$$

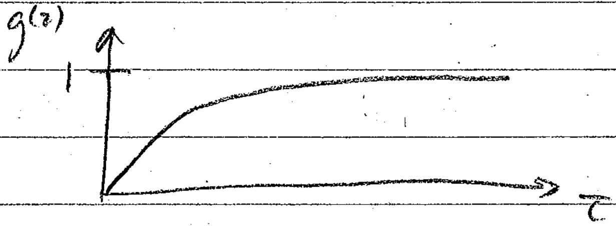
$$g^{(2)}(0) = 0 \qquad \text{because } \hat{\pi}(t)\hat{\pi}(t) = 0$$

<u>Sub-Poissonian!</u>

How calculate $\langle \hat{\pi}^+(t)\hat{\pi}^+(t+\tau)\hat{\pi}(t+\tau)\,\hat{\pi}(t)\rangle$

<u>Assume:</u> After a long enough time delay $\tau$, $\hat{\pi}^\dagger(t+\tau)\hat{\pi}(t+\tau)$ and $\hat{\pi}^\dagger(t)\hat{\pi}(t)$ become <u>independent</u> operators, and

$$g^{(2)}(\tau) \to \frac{\langle \hat{\pi}^+(t)\hat{\pi}(t)\rangle\langle \hat{\pi}^+(t+\tau)\hat{\pi}(t+\tau)\rangle}{\langle \hat{\pi}^\dagger(t)\hat{\pi}^+(t)\rangle^2} = 1 \quad \text{for } \tau \to \infty$$




<u>Anti-bunched.</u>

Loudon applies a mixture of OBE and other things to arrive at

$$g^{(2)}(\tau) = 1 - e^{-2\gamma_{sp}|\tau|}$$