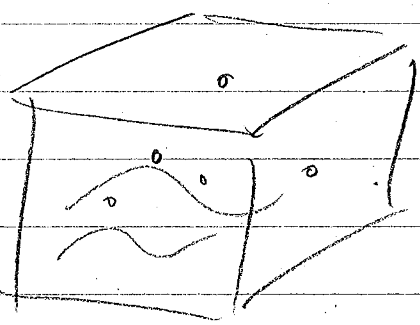
13. Amplification & attenuation; lasers

- Consider a system of interacting atoms + light



N atoms	The light:
Two levels $\hbar\omega_0 = E_2 - E_1$	Frequency ω
Linewidth: $\gamma = \gamma_{sp} + \gamma_{coll}$	Narrow bandwidth
$\gamma_{coll} \gg \gamma_{sp}$	$\approx$ single mode.

Our goal: Rate equations for atoms and light

- Rate eq for atoms
(Basically Einstein theory):

$$\frac{dN_2}{dt} = -\Gamma_{sp} N_2 - n\Gamma_{st} N_2 + n\Gamma_{st} N_1 = -\frac{dN_1}{dt}$$

We know $\Gamma_{sp} = A_{21} = 2\gamma_{sp}$
[We're just giving it a new name for this chapter!]
$\leftrightarrow$ rate of emission into all light modes (summed).

What is Γ_{st}, the rate of stimulated abs & em?

Einstein theory:

$$n \Gamma_{st} = B_{12} \langle W(\omega) \rangle$$

energy of radiation per unit volume & unit frequency

$$= B_{12} \frac{n \hbar \omega}{V} \frac{\gamma/\pi}{(\omega_0 - \omega)^2 + \gamma^2}$$

$$= \{\text{specialize to } \omega = \omega_0\}$$

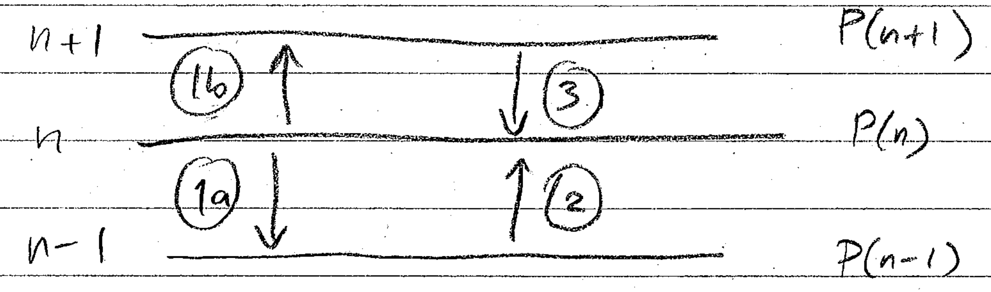
$$= B_{12} \frac{\hbar \omega}{\pi V \gamma} \cdot n$$

$$\Gamma_{st} = \frac{2\pi c^3}{V \omega_0^2} \frac{\gamma_{sp}}{\gamma} \qquad (\text{remember } \gamma = \gamma_{coll} + \gamma_{sp})$$

- Here, we have inserted a statistical expression due to the collision broadening.
  Could not hope to solve the N-body problem exactly - have to insert Lorentzian by hand!

- Now, rate equation for radiation?

Work with probabilities $P(n)$



(1a) Absorption, $-N_1 \Gamma_{st} \cdot n \cdot P(n)$

(1b) Emission, $-N_2 \Gamma_{st} \cdot (n+1)\, P(n)$

This is for spontaneous emission into this mode only.

In contrast, Γ_{sp} gives spont. em. into all modes, summed and $\Gamma_{sp} \neq \Gamma_{st}$.

(2) Emission $N_2 \Gamma_{st}\, n P(n-1)$

(3) Absorption $N_1 \Gamma_{st} \cdot (n+1) \cdot P(n+1)$

Put all this together to get the total rate

$$\frac{d}{dt} P(n) = \Gamma_{st} \Big\{ N_2 n P(n-1) - N_1 n P(n) - N_2 (n+1) P(n) + N_1 (n+1) P(n+1) \Big\}$$

Mean number of photons

$$\langle n \rangle = \sum_n n P(n)$$

$$\Rightarrow \frac{d\langle n \rangle}{dt} = \sum_n n \frac{dP(n)}{dt}$$

$\frac{dP(n)}{dt}$ depend on N_1 and N_2

$\frac{dN_1}{dt}$ and $\frac{dN_2}{dt}$ depend on n

$\Rightarrow$ difficult!

Full equation should be solved with $N_2(n) \neq N_2(n+1)$ etc being, in general, different.

Simplification if N_1 & N_2 do not depend strongly on n: [Exercise!]

$$\boxed{\frac{d\langle n\rangle}{dt} = \Gamma_{st}\left\{N_2 - (N_2 - N_1)\langle n\rangle\right\}} \qquad (*)$$

★ Fixed atomic populations

Suppose N_1 and N_2 are kept fixed (by some external means)

① First look at steady state:

$$0 = \frac{dP(n)}{dt} = \Gamma_{st}\left\{N_2\, n\, P(n-1) - N_1\, n\, P(n) - N_2 (n+1) P(n) + N_1 (n+1) P(n+1)\right\}$$

$(n=0)$ $\quad -N_2 P(0) + N_1 P(1) = 0$

$$\Rightarrow \quad P(1) = \frac{N_2}{N_1} P(0)$$

... and so on (induction) ...

find $\quad P(n) = \left(\frac{N_2}{N_1}\right)^n P(0)$

Normalization $\quad \sum_{n=0}^{\infty} P(n) = 1$

$$1 = \sum_{n=0}^{\infty} \left(\frac{N_2}{N_1}\right)^n P(0) = \{\text{geometric}\} =$$

$$= \frac{1}{1 - \frac{N_2}{N_1}} \cdot P(0) \quad \Rightarrow \quad P(0) = 1 - \frac{N_2}{N_1}$$

$$\therefore \boxed{P(n) = \left(1 - \frac{N_2}{N_1}\right)\left(\frac{N_2}{N_1}\right)^n}$$

Steady state fixed N_j

obtain

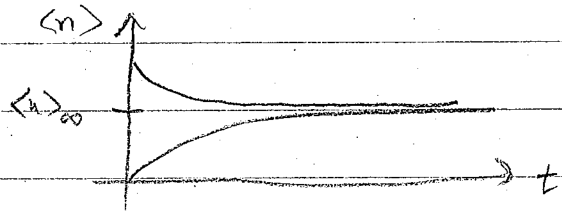
$$\langle n\rangle^2\left[g^{(2)}(0)-2\right]=\langle n\rangle_0^2\left[g_0^{(2)}(0)-2\right]e^{-2\Gamma_{st}(N_1-N_2)t}$$

<u>Note:</u> $\left.\begin{array}{l} g^{(2)}(\tau) \doteq g^{(2)}(t,t+\tau) \\ g^{(2)}(0) = g^{(2)}(t,t) \end{array}\right\}$ is the correlation for <u>time delay</u> τ, after time t

$\left.\begin{array}{l} g_0^{(2)}(\tau) = g^{(2)}(0,\tau) \\ g_0^{(2)}(0) = g^{(2)}(0,0) \end{array}\right\}$ is the correlation for time delay τ, initially

<u>This means:</u>

(+) <u>Mean number of photons</u>
initial value: arbitrary $\langle n\rangle_0$
approaches final value $\langle n\rangle_\infty$ exponentially



\- <u>Second-order coherence</u>
Initial value: arbitrary $g_0^{(2)}(0)$
approaches final value: $g_\infty^{(2)}(0)=2$ exponentially

... so this process <u>creates chaotic light</u>.

<u>Note:</u> If $N_2 > N_1$, then $\langle n\rangle$ increases without bound & $g^{(2)}(0)$ may decrease.
This is <u>population inversion</u> – necessary for a laser!

... so chaotic light is <u>created</u> <u>in time</u>.

Single-mode laser theory

Necessary for laser: rate of <u>stimulated</u> emission > absorption!

(<u>Stim.</u> yields more photons into the <u>same</u> mode
Spont. yields photons in all possible modes.
-generates chaotic light.)

From ch. 1: $N_2 B\langle W\rangle > N_1 B\langle W\rangle$

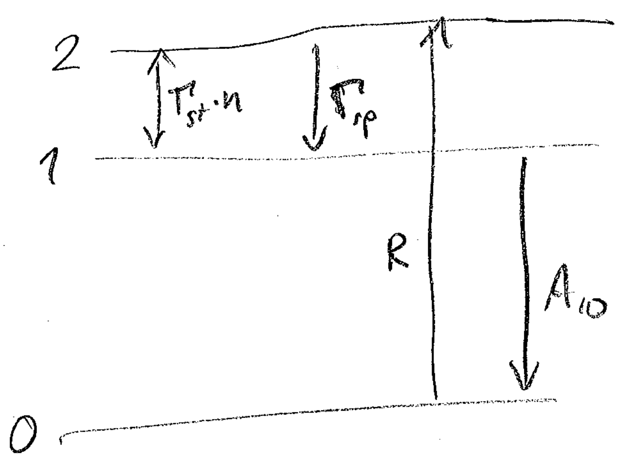
$$\Rightarrow N_2 > N_1$$

which is <u>impossible</u> in equilibrium.

<u>Need:</u> <u>Population inversion</u>

= by some means, make $N_2 > N_1$.

<u>Idea:</u> Need <u>three</u> levels



$A_{10} \gg R, \Gamma_{sp}, \Gamma_{st}\cdot n$

this will give population inversion

"Atom spend little time in level 1 because short lifetime"

Steady-state: $\frac{d}{dt} P(n) = 0$

$$\Rightarrow N_2 = \frac{NR}{\Gamma_{sp} + \Gamma_{st}\, n}$$

We will see that: n in denominator $\Longleftrightarrow$ coherent light!

Laser amplification

A laser is characterized by a "cooperation parameter"

$$C = \frac{NR\,\Gamma_{st}}{\Gamma_{sp}\,\Gamma_{cav}}$$

Many atoms
High pump rate
High stimulation rate
} good!

High loss rate
High spont. emission rate
} Bad!

Have

$$\Gamma_{st}\Gamma_{cav}\langle n\rangle^2 - \left(NR\,\Gamma_{st} - \Gamma_{sp}\Gamma_{cav}\right)\langle n\rangle - NR\,\Gamma_{st} = 0$$

Define $n_s = \dfrac{\Gamma_{sp}}{\Gamma_{st}}$ "the saturation photon number"

$$\Rightarrow \langle n\rangle^2 - (C-1)\, n_s \langle n\rangle - C n_s = 0$$

$$\Rightarrow \langle n\rangle = \tfrac{1}{2}(C-1)\, n_s + \tfrac{1}{2}\sqrt{(C-1)^2 n_s^2 + 4 C n_s}$$

(fig. 7.5)