III. Multimode quantum optics

- Discrete modes – simple!

Ex. Number states $|\{n_{k\lambda}\}\rangle$ as before

Coherent states $|\{\alpha\}\rangle = |\alpha_{k_1}\alpha_{k_2}\alpha_{k_3}\cdots\rangle$

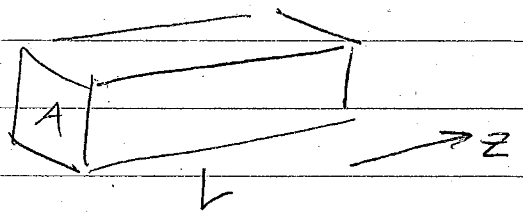
One α for each mode.

More interesting: Continuous modes

Every real pulse is finite in time & length: superposition of several modes with different freq.

Assume all modes along the z-direction & same polarization (drop unnecessary complications!)

Geometry $A \times L$



Spacing between wavenumbers $\Delta k = \frac{2\pi}{L}$

frequencies $\Delta\omega = \frac{2\pi c}{L}$

Go to continuous case: $L \to \infty$

$$\hat{a}_k \to \sqrt{\Delta\omega}\,\hat{a}(\omega)$$

$$\sum_k \to \frac{1}{\Delta\omega}\int d\omega$$

Obtain

$$[\hat{a}(\omega), \hat{a}^{+}(\omega')] = \delta(\omega - \omega')$$

$$\hat{E}_T^{+}(z,t) = i \int_0^{\infty} d\omega \sqrt{\frac{\hbar\omega}{4\pi\varepsilon_0 c A}}\, \hat{a}(\omega) e^{-i\omega(t - z/c)}$$

($\hat{B}^{+}$ in London 6.2.7; $\hat{E}_T^{-}$ and $\hat{B}^{-}$ are conjugates)

Hamiltonian

$$\hat{H}_R = \int_0^{\infty} d\omega\, \hbar\omega\, \hat{a}^{+}(\omega)\hat{a}(\omega) + \text{vacuum energy}$$

↑ not really well-defined...

Number operator

$$\hat{n} = \int d\omega\, \hat{a}^{+}(\omega)\hat{a}(\omega)$$

Number states $|1_\omega\rangle = \hat{a}(\omega)|0\rangle$
normalized as

$$\langle 1_{\omega'} | 1_\omega \rangle = \delta(\omega - \omega')$$

- Work as a function of time instead of frequency!

Def

$$\boxed{\hat{a}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, \hat{a}(\omega) e^{-i\omega t}}$$

$$\Rightarrow [\hat{a}(t), \hat{a}^{+}(t')] = \delta(t - t')$$

and

$$\hat{n} = \int_{-\infty}^{\infty} dt\, \hat{a}^{+}(t)\hat{a}(t)$$

Define <u>flux operator</u>

$$\hat{f}(t) = \hat{a}^{+}(t)\,\hat{a}(t)$$

the flux of photons through the area A

so that the Poynting vector

$$\hat{I}(z,t) = \frac{1}{A}\hbar\omega\,\hat{f}(t - z/c)$$

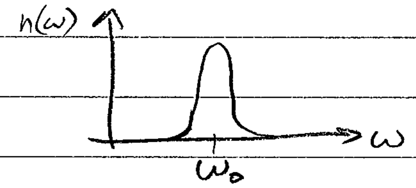
Then the mean photon number is going to be

$$\langle n \rangle = \int dt \langle f(t) \rangle$$

E-field operator

$$\hat{E}_T^{+}(z,t) = i\int_0^{\infty} d\omega \sqrt{\frac{\hbar\omega}{4\pi\varepsilon_0 cA}}\,\hat{a}(\omega)\,e^{-i\omega(t - \frac{z}{c})}$$

If we are going to consider <u>narrow</u>-band light, so that we expect



then we may approximate

$$\hat{E}_T^{+}(z,t) = i\sqrt{\frac{\hbar\omega_0}{4\pi\varepsilon_0 cA}}\int_{-\infty}^{\infty} d\omega\,\hat{a}(\omega)\,e^{-i\omega(t - z/c)}$$

$$= i\sqrt{\frac{\hbar\omega_0}{4\pi\varepsilon_0 cA}}\,\hat{a}(t - z/c)$$

this means: We <u>anticipate</u> that we will work only with states where a narrow range of frequencies is excited.
Otherwise, must use <u>full</u> expression for E

Then the coherences are

$$g^{(1)}(\tau) = \frac{\langle \hat{a}^{+}(t)\,\hat{a}(t+\tau)\rangle}{\langle \hat{a}^{+}(t)\,\hat{a}(t)\rangle}$$

$$g^{(2)}(\tau) = \frac{\langle \hat{a}^{+}(t)\,\hat{a}^{+}(t+\tau)\,\hat{a}(t+\tau)\,\hat{a}(t)\rangle}{\langle \hat{a}^{+}(t)\,\hat{a}(t)\rangle^{2}}$$ (Narrow band)

- Wave packets

The photons created in realistic experiments by some emission process are in the form of a wave packet

[Cascade emission: London p. 220 §5.8]

Ex: A Gaussian wavepacket

Spectral amplitude (frequency distribution)

$$\xi(\omega) = \frac{1}{(2\pi\Delta^2)^{1/4}} e^{-i(\omega_0-\omega)t_0 - \frac{(\omega_0-\omega)^2}{4\Delta^2}}$$

centered around ω_0 with width Δ.

t_0 is just a constant [will soon see significance].

Fourier transform is

$$\xi(t) = \left(\frac{2\Delta^2}{\pi}\right)^{1/4} e^{-i\omega_0 t - \Delta^2(t-t_0)^2}$$

Use these functions to define the state

$$|n_\xi\rangle = (\hat{a}_\xi^{+})^n |0\rangle / \sqrt{n!}$$

where $\hat{a}_\xi^{+} = \int d\omega\, \xi(\omega)\, \hat{a}^{+}(\omega) = \int dt\, \xi(t)\, \hat{a}^{+}(t)$

photon wave packet creation operator.

(In reality, the common cases are $n=1$ or 2)

[Of course, could do the same for *any* function $\xi(\omega)$.]

We find $[\hat{a}_\xi, \hat{a}_\xi^+] = 1$

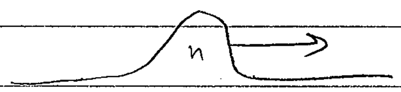
Properties of $|n_\xi\rangle$:

$$\langle n_\xi | \hat{E}_T | n_\xi \rangle = 0$$

as usual for number states.

$$\langle n_\xi | \hat{f}(t) | n_\xi \rangle = n |\xi(t)|^2 \quad \text{photon flux}$$

so $\xi(t)$ describes the amplitude of the wave packet



Note that

$$\hat{I}(z,t) = \sqrt{\frac{2}{\pi}} \frac{n\hbar\omega_0\Delta}{A} e^{-2\Delta^2(t-t_0-z/c)^2}$$

Continuous-mode coherent states

Given a spectral function $\alpha(\omega)$, $\alpha(t)$, define

$$|\alpha\rangle = e^{\hat{a}_\alpha^+ - \hat{a}_\alpha}|0\rangle$$

where now $\int d\omega |\alpha(\omega)|^2 = \int dt |\alpha(t)|^2 = \langle n \rangle$

$$\Rightarrow [\hat{a}_\alpha, \hat{a}_\alpha^+] = \langle n \rangle$$

$|\alpha\rangle$ is second-order coherent at all space-time points.

The output from a laser is often described by $|\alpha\rangle$

Photon bunching

Second-order coherence:

Chaotic light	$g^{(2)}(\tau) = 1 + e^{-2\gamma\|\tau\|}$	$\Rightarrow g^{(2)}(0) = 2$
Coherent state	$g^{(2)}(\tau) = 1$	$g^{(2)}(0) = 1$
Number state	$g^{(2)}(\tau) = 1 - \frac{1}{n}$	$g^{(2)}(0) = 1 - \frac{1}{n}$

Photon bunching: $g^{(2)}(0) > 1$

"If I measure one photon, the probability of detecting another just after is higher than on average".

~"photons come in bunches"

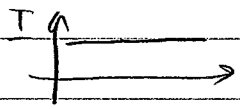
Cartoons in London p. 250 Fig. 6.1

Classically:

Coherent light: $g^{(2)}(0) = 1$

At any time instant, the probability is the same for detecting a photon.

Classically:



stable wave

Anti-bunching: $g^{(2)}(0) < 1$

After detection of one photon, there is lower probability than average to detect another one. Typical when photons come with regular time intervals between them.

Cannot happen for any classical field.