(1) Planck's radiation law & Einstein coefficients

26/3 13-15 | 1.1 - 1.5

$\vec{E}, \vec{B}$

Consider an electromagnetic field in vacuum

Maxwell's eqs $\Rightarrow$

$$\nabla^2 \vec{E}(\vec{r},t) = \frac{1}{c^2} \frac{\partial^2 \vec{E}(r,t)}{\partial t^2}$$

$$\nabla \cdot \vec{E} = 0$$

& the value of $\vec{B}$ follows from

$$\nabla \times \vec{E}(r,t) = -\frac{\partial \vec{B}}{\partial t}$$

Assume box with conducting walls $0 \leq x, y, z \leq L$

$\Rightarrow$ tangential components of $\vec{E}$ vanish

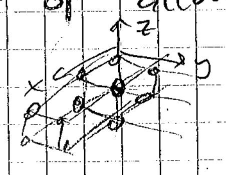
Sol: $\vec{E} = (E_x, E_y, E_z)$ where

$$\begin{cases} E_x(\vec{r},t) = E_x(t)\cos(k_x x)\sin(k_y y)\sin(k_z z) \\ E_y(\vec{r},t) = E_y(t)\sin(k_x x)\cos(k_y y)\sin(k_z z) \\ \quad z \qquad\qquad z \qquad \sin \qquad \sin \qquad \cos \end{cases}$$

& $\vec{k} = (k_x, k_y, k_z)$ with $k_x = \frac{\pi}{L} m_x$, $m_x = 0, 1, 2, 3, \dots$

etc.

$\Rightarrow$ lattice of allowed wavevectors



(see Fig. 1.2)

Furthermore, M.E. $\Rightarrow$ $\vec{k} \cdot \vec{E}(t) = 0$

where $\vec{E}(t) = (E_x(t), E_y(t), E_z(t))$

is the space-independent factor

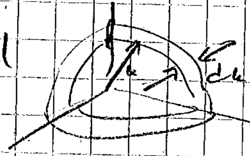
For each $\vec{k}$, there are 2 linearly independent directions for $\vec{E}(t)$, $\vec{\mathcal{E}}_{\vec{k}\lambda}(t)$ with $\lambda = 1, 2$

Each choice of $(\vec{k}, \lambda)$ is one **field mode**.

Frequency + direction of propagation + polarization

★ How many field modes between k, $k+dk$

One octant of spherical shell



$$\frac{1}{8}\; \underbrace{4\pi k^2 dk}_{\text{shell volume}} \; \underbrace{\left(\pi/L\right)^{-3}}_{\#\ k\text{-values per unit volume}} \cdot 2$$

octant   polarization

Density of modes per unit volume

$$\boxed{\rho(k)\,dk = \frac{k^2 dk}{\pi^2}}$$

By using $\omega = ck$ we obtain

$$\rho(\omega)\,d\omega = \frac{\omega^2 d\omega}{\pi^2 c^3}$$

Planck's quantization hypothesis

Classically, the total energy

$$\mathcal{E} = \frac{1}{2}\int dV \left(\varepsilon_0 \vec{E}(r,t)^2 + \frac{1}{\mu_0} \vec{B}(r,t)^2 \right)$$

$(\varepsilon_0 \mu_0 = c^2)$

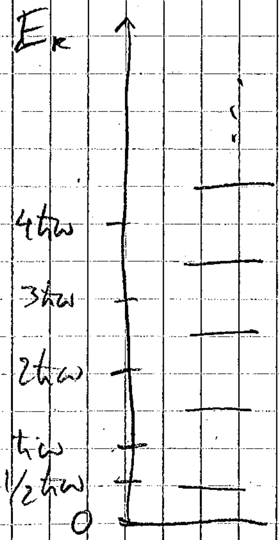
and $\vec{E}_0(t)$ can take on any continuous value.

→ Instead, Planck suggested
energy of each field mode can have
any of the discrete values

$$\mathcal{E}_k = (n + \tfrac{1}{2})\hbar\omega \quad n = 0, 1, 2, 3, \ldots$$

zero-point energy!

n is the "number of photons" in each field mode
⇒ energy in a field mode can only change in
discrete steps - quanta



allowed energies

$n=0$: $\mathcal{E}_k = \frac{1}{2}\hbar\omega$ the zero-point energy. Even though the field mode is empty.

$n>0$: The field mode is "excited with n photons". When n changes, photons are created or destroyed.

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Planck's radiation law

Assume a cavity in thermal equilibrium.

How many photons with frequency ω are excited at temperature T ?

Probability that a mode has energy E_n is given by the Boltzmann factor

$$P(n) = \frac{e^{-E_n/k_BT}}{\sum_n e^{-E_n/k_BT}} \qquad E_n = (n+\tfrac{1}{2})\hbar\omega$$

$$= \frac{e^{-n\frac{\hbar\omega}{k_BT}}}{\sum_n e^{-n\frac{\hbar\omega}{k_BT}}} \qquad \left(\tfrac{1}{2}\hbar\omega \text{ cancels out}\right)$$

$$= \frac{U^n}{\sum_{n=0}^{\infty} U^n} \qquad \text{where } U = e^{-\frac{\hbar\omega}{k_BT}}$$

$$\{\text{geom}\} = U^n(1-U)$$

Mean number of photons in this mode

$$\langle n \rangle = \sum_{n=0}^{\infty} n P(n) = (1-U)\sum_{n=0}^{\infty} n U^n = (1-U)U\frac{\partial}{\partial U}\sum_n U^n$$

$$= (1-U)U\frac{\partial}{\partial U}\frac{1}{1-U} = \frac{U}{1-U}$$

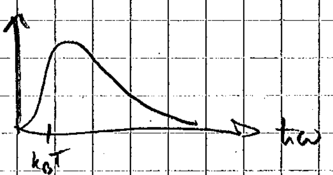
$$= \frac{1}{e^{\hbar\omega/k_BT}-1} \qquad \{\text{The Bose-Einstein dist. fcn}\}$$

Calculate the <u>energy density</u> of the radiation in the cavity

$$\langle W_T(\omega)\rangle d\omega = \{\#\text{ modes}\}\cdot\left\{\#\begin{matrix}\text{photons}\\ \text{per mode}\end{matrix}\right\}\cdot\left\{\begin{matrix}\text{energy}\\ \text{per photon}\end{matrix}\right\}$$

$$= \frac{\omega^2}{\pi^2 c^3} d\omega \frac{1}{e^{\hbar\omega/k_BT}-1}\hbar\omega$$

$$= \frac{\hbar\omega^3}{\pi^2 c^3}\frac{1}{e^{\hbar\omega/k_BT}-1}d\omega$$



The black-body spectrum.

Planck 1900: the quantization hypothesis was only way to reproduce the empirically measured spectrum.

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Fluctuations

Photons are continually created & destroyed. Even if $\langle n \rangle$ = constant in equilibrium, there are still fluctuations.

<u>Ergodicity</u>: A time average is the same as the average taken over a large number of identical systems (ensemble average)

Write
$$\langle n \rangle = \frac{U}{1-U}$$

$$\Rightarrow U = \frac{\langle n \rangle}{1+\langle n \rangle}$$

then
$$P(n) = \frac{\langle n \rangle^n}{(1+\langle n \rangle)^{1+n}}$$

Variance of photon number

$$(\Delta n)^2 = \langle n^2 \rangle - \langle n \rangle^2 = \sum_n (n - \langle n \rangle)^2 P(n)$$

Standard deviation

$$\Delta n = \sqrt{\langle n^2 \rangle - \langle n \rangle^2}$$

can be shown to be (book)

$$\Delta n = \sqrt{\langle n \rangle^2 + \langle n \rangle}$$

<u>Note</u>: 1) If $\langle n \rangle \gg 1 \Rightarrow \Delta n \approx \langle n \rangle$

2) Always $\Delta n > \langle n \rangle$, <u>Wide-spread</u> distribution.

Fig 1.7

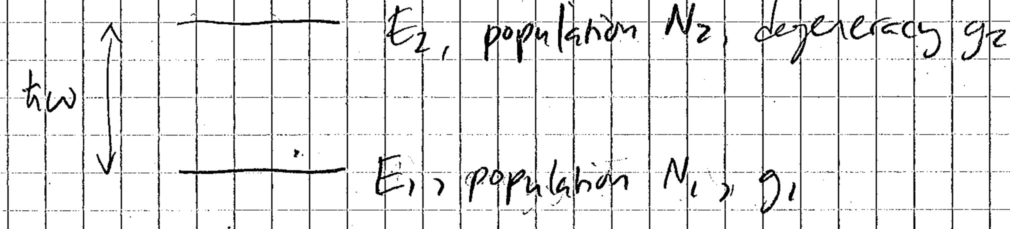
Einstein's theory of absorption & emission

N atoms interacting with an EM field.

A lot can be done without applying a fully QM theory, just 2 assumptions:

1) Atoms have quantized energy levels
2) Light is created & destroyed in quanta.

Assume N identical atoms
with two energy levels
(only these two matter - explain later)



assume $E_2 - E_1 = \hbar\omega$

and $N_1 + N_2 = N$

The density of radiation energy

$$\langle W(\omega)\rangle = \underbrace{\langle W_T(\omega)\rangle}_{\text{thermal}} + \underbrace{\langle W_E(\omega)\rangle}_{\text{external source (beam of light)}}$$

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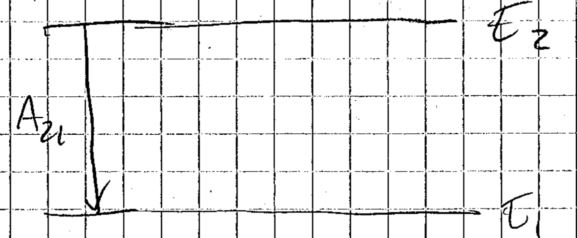
- Energy exchange:



Spontaneous emission rate A_{21} $[s^{-1}]$
prob. per unit time

Absorption $\langle W(\omega)\rangle B_{12}$ $[s^{-1}]$

Stimulated emission $\langle W(\omega)\rangle B_{21}$ $[s^{-1}]$

the "Einstein coefficients".

Rate equations for the level populations

$$\frac{dN_1}{dt} = N_2 A_{21} - N_1 B_{12}\langle W(\omega)\rangle + N_2 B_{21}\langle W(\omega)\rangle$$

$$\frac{dN_2}{dt} = -\frac{dN_1}{dt}$$

Steady state solution:

Put $\frac{dN_1}{dt} = \frac{dN_2}{dt} = 0$

$$N_2 A_{21} - N_1 B_{12}\langle W(\omega)\rangle + N_2 B_{21}\langle W(\omega)\rangle$$

Now assume thermal equilibrium: $\langle W_E(\omega)\rangle = 0$

We can calculate the Einstein coeffs.

$$\langle W_T(\omega)\rangle \left(B_{21} N_2 - B_{12} N_1\right) = -A_{21} N_2$$

$$\langle W_T(\omega)\rangle = \frac{A_{21}}{\frac{N_1}{N_2} B_{12} - B_{21}}$$

Use thermal equ. => Boltzmann distribution

$$\frac{N_1}{N_2} = \frac{e^{-E_1/k_BT}}{e^{-E_2/k_BT}} \cdot \frac{g_1}{g_2} = \frac{g_1}{g_2} e^{\hbar\omega/k_BT}$$

also use Planck's law

$$\langle W_T(\omega)\rangle = \frac{\hbar\omega^3}{\pi c^3} \frac{1}{e^{\hbar\omega/k_BT} - 1}$$

$$\frac{\hbar\omega^3}{\pi c^3} \frac{1}{e^{\hbar\omega/k_BT} - 1} = \frac{A_{21}}{\frac{g_1}{g_2} e^{\frac{\hbar\omega}{k_BT}} B_{12} - B_{21}}$$

this holds for all temperatures T if

$$\begin{cases} \frac{g_1}{g_2} B_{12} = B_{21} \quad \Rightarrow \quad g_1 B_{12} = g_2 B_{21} \\ \frac{A_{21}}{B_{21}} = \frac{\hbar\omega^3}{\pi c^3} \end{cases}$$