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CHAPTER 18

The EPR paradox, hidden variables, and Bell's theorem

Quantum mechanics is an immensely successful theory, occupying a unique position in the history of science. It has solved mysteries ranging from macroscopic superconductivity to the microscopic theory of elementary particles and has provided deep insights into the nature of vacuum on the one hand and the description of the nucleon on the other. Whole new fields such as quantum optics and quantum electronics owe their very existence to this body of knowledge.

However, despite the stunning successes of quantum mechanics, there is no general agreement on the conceptual foundations and interpretation of the subject. The theory provides unambiguous information about the outcome of a measurement of a physical object. However, many feel that it does not provide a satisfactory answer to the nature of the "reality" we should attribute to the physical objects between the acts of measurement.

The conceptual difficulty comes about because the wave function $|\psi\rangle$ is usually given by a coherent superposition of various distinguishable experimental outcomes. If we denote the collection of states that represent the possible outcomes of an experiment by $|\psi_j\rangle$, then $|\psi\rangle = \sum_j c_j|\psi_j\rangle$ where $c_j = \langle\psi_j|\psi\rangle$. The probability of the outcome $|\psi_j\rangle$ is $p_j = |c_j|^2$. In the process of measurement, the so called *collapse of the wave function* takes place and a single, definite state $|\psi_i\rangle$ of the physical object is chosen. The difficulty comes about in the interpretation of the mechanism by which this definite state is chosen from amongst all the possible outcomes.

An important consequence of the quantum mechanical formalism is that it does not seem to allow a *local* description of events in the sense discussed below. Alternatively, a *local* theory can be achieved but with the additional difficulties of *negative* probabilities.

This counter-intuitive nonlocal aspect of quantum mechanics has been a subject of debate since the early days. In particular, Einstein, Podolsky, and Rosen (EPR) conjectured, on the basis of a gedanken experiment, that quantum mechanics is an incomplete theory. In the absence of a concrete experimental situation to test the *reality* and *locality* aspects of quantum mechanics, the debate concerning the foundations of quantum mechanics continued to be essentially philosophical in nature for many years.

The situation however changed dramatically when, in 1964, J. S. Bell formulated certain inequalities, known as Bell's inequalities,* which should always be true for any theory that satisfies the *intuitively reasonable* notions of reality and locality. One of the most interesting results of modern physics is that quantum mechanics violates Bell's inequalities in certain situations, and that experimental results agree with the quantum mechanical predictions.

In this chapter, we present the EPR arguments concerning the *incompleteness* of quantum mechanics. We then discuss Bell's inequality and the quantum mechanical results violating it. The disagreement between Bell's inequality and the quantum mechanical predictions is further sharpened by the study of various alternative theories to quantum mechanics, hidden variable (HV) theories being prominent among these. In order to better understand the problem, we show that a 'nonlocal' hidden variable theory can be developed which is in agreement with quantum theory. Finally, we show that a new kind of equality, the so-called Greenberger–Horne–Zeilinger (GHZ) equality, is violated by quantum mechanics.

The present chapter, and the next two chapters as well, deal with interpretational problems of quantum mechanics. In all such studies, we follow the lead of Lamb [1969]; namely, develop the analysis around the theory for an apparatus which is designed to make the appropriate measurements. This sharpens the arguments and keeps the goal in focus.

18.1 The EPR 'paradox'

In 1935, Einstein, Podolsky, and Rosen (EPR) presented an argument to show that there are situations in which the general probabilistic scheme of quantum theory seems to be incomplete. Here we present a variation of this argument due to Bohm.

* For a beautiful account of the subject, see Mermin [1990a,b].

Let us consider a two-component system consisting of two spin-1/2 particles (e.g., the Hg_2 molecule). Up to some time $t = 0$, these particles are taken to be in a bound state of zero angular momentum. We designate the corresponding state vector as $|\Psi_{1,2}\rangle$. At time $t = 0$ we 'turn off' the binding potential (e.g., we photo-disintegrate the molecule) but introduce no angular momentum into the system and do not disturb the spins in any way. The separate parts of the system are now free to move off to opposite sides of the laboratory (or the universe for that matter). We now consider two kinds of experimental arrangements as shown in Figs. 18.1(a) and 18.1(b).

First we consider the case where we measure the z-component of spin 1 as indicated in Fig. 18.1(a). Before making the measurement on spin 1 the state vector for the system is

$$|\Psi_{1,2}\rangle = \frac{1}{\sqrt{2}}(|\uparrow_1, \downarrow_2\rangle - |\downarrow_1, \uparrow_2\rangle), \tag{18.1.1}$$

where $|\uparrow_1, \downarrow_2\rangle$ labels the state of particle 1 with spin projection $+1/2$, and the state of the second particle with spin projection $-1/2$ with respect to the z-axis, etc.

Now one version of the EPR argument runs as follows:

(1) Pick an arbitrary direction, which we can take to be the z-axis, and pass one Hg atom (say atom 1) through a Stern–Gerlach apparatus (SGA) oriented along the z-axis. The particle will now be deflected in either the $+$ or $-z$ direction, say $+z$. Thus we know the value of σ_z of that state is $+1$.

(2) Knowing that the spin of particle 1 is up, we now know the spin of particle 2 is down. But if we then pass atom 2 through a SGA oriented along the x-axis we will find that particle 2 has a definite spin along the x-direction (either $+x$ or $-x$), i.e., we now know the value of σ_x.

(3) Therefore, as the argument goes, we know both the z and x components of spin 2, which is a violation of complementarity.

It is worthwhile to restate the above version of the EPR paradox, which focused on an apparent violation of complementarity (we have "found" both σ_x and σ_z), in terms of a state vector picture.

Let us consider first the case where we measure the z-component of spin 1 as indicated in Fig. 18.1(a). Before making the measurement (at some time t_0) on spin 1 the state vector for the system is

$$|\psi^{<}_{1,2}\rangle = \frac{1}{\sqrt{2}}(|\uparrow_1, \downarrow_2\rangle - |\downarrow_1, \uparrow_2\rangle), \tag{18.1.2}$$

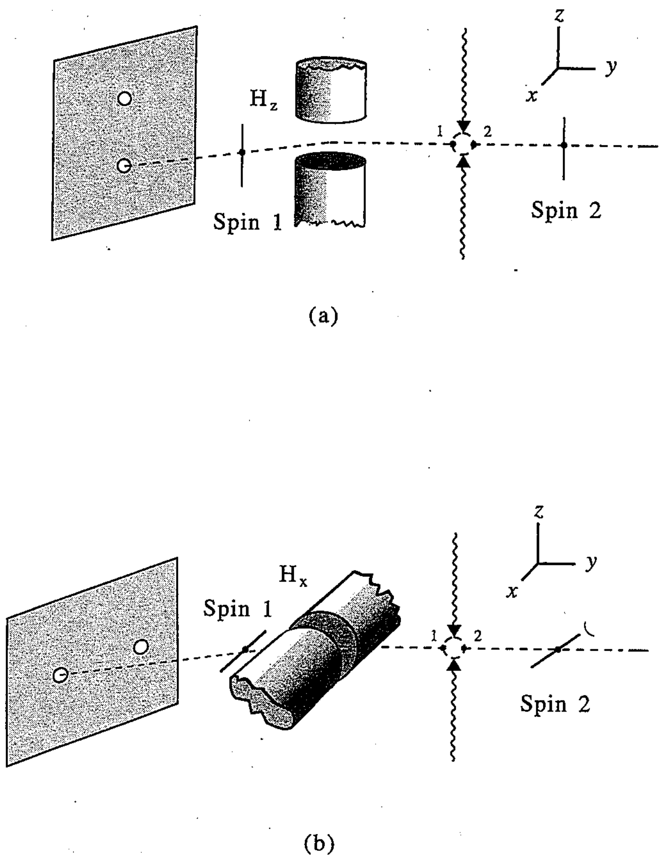

Fig. 18.1 Schematic of the EPR gedanken experiment. A spin-zero system such as orthohydrogen is split by an external field. The two spin-1/2 particles (protons) proceed in opposite directions. Particle 1 passes through a Stern-Gerlach apparatus (a) oriented along the z-axis and (b) oriented along the x-axis.

where $\psi^<$ denotes the state *before* ($t < t_0$) 'looking' at particle 1. Now if after a measurement on particle 1 we find it to be, say, in the spin down state $|\downarrow_1\rangle$, then the state of particle 2 is given by

$$|\psi_2^>\rangle = |\uparrow_2\rangle. \tag{18.1.3}$$

Here $\psi_2^>$ denotes the state of system 2 *after* ($t > t_0$) measuring particle 1.

At this point EPR argue as follows: since at the time of measurement the two systems no longer interact, no real change has taken place in the second system as a consequence of anything that may

happen to the first system. That is, there exists no interaction between the two systems. Furthermore, EPR argue, since we have not affected particle 2 by looking at 1, the state of particle 2 must be the same before and after the measurement. That is,

$$|\psi_2^<\rangle = |\psi_2^>\rangle = |\uparrow_2\rangle, \tag{18.1.4}$$

where $|\psi_2^<\rangle$ and $|\psi_2^>\rangle$ denote the 'before' and 'after' states.

But we could have just as well decided to measure the x-component of particle 1 as in Fig. 18.1(b). Therefore, we naturally describe our spins in terms of $|\pm x\rangle$ states

$$|\pm x\rangle \equiv |\pm\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \pm |\downarrow\rangle), \tag{18.1.5}$$

and our spin singlet state before the measurment is

$$|\phi_{1,2}^<\rangle = \frac{1}{\sqrt{2}}(|+_1 -_2\rangle - |-_1 +_2\rangle). \tag{18.1.6}$$

Now after finding spin 1 to be in, say, the state $|-_1\rangle$, we have

$$|\phi_2^>\rangle = |+_2\rangle, \tag{18.1.7}$$

which, following the EPR argument as before, implies

$$|\phi_2^<\rangle = |+_2\rangle, \tag{18.1.8}$$

a very unsatisfactory state of affairs! For in the words of EPR: 'Thus, *it is possible to assign two different state vectors* [in our notation $|\uparrow_2\rangle$ and $|+_2\rangle$] *to the same reality*.'

One way out of this problem is to argue that when we are looking at a subsystem (e.g., particle 2 only), then we should be using a density matrix formulation. In general, when we are considering a composite system consisting of two subsystems, A and B, and if we are only interested in expectation values of operators $\hat{Q}_A$ which refer to system A alone, i.e.,

$$\hat{Q} = \hat{Q}_A \otimes 1_B, \tag{18.1.9}$$

then we are led to introduce the reduced density matrix ρ_A. That is, expressed in terms of the total density matrix $\rho_{A,B}$ we have

$$\begin{aligned}\langle Q\rangle &= \mathrm{Tr}_{A,B}(\rho_{AB}\hat{Q}) = \sum_{a,b}\langle a,b|\rho_{AB}\hat{Q}|a,b\rangle \\ &= \sum_a \langle a|\sum_b \langle b|\rho_{AB}1_B|b\rangle \hat{Q}_A|a\rangle = \mathrm{Tr}_A(\rho_A\hat{Q}_A),\end{aligned} \tag{18.1.10}$$

where reduced density matrix for system A is

$$\rho_A^{(r)} = \sum_b \langle b|\rho_{AB} 1_B|b\rangle = \mathrm{Tr}_B(\rho_{AB}). \tag{18.1.11}$$

Hence we should properly be considering the density matrix for system 2 (before looking at 1). In the first experiment, this is given by

$$\begin{aligned}\rho_2^{<}(I) &= \mathrm{Tr}_1[\rho_{12}^{<}(I)] = \langle\uparrow_1|\psi_{12}^{<}\rangle\langle\psi_{12}^{<}|\uparrow_1\rangle + \langle\downarrow_1|\psi_{12}^{<}\rangle\langle\psi_{12}^{<}|\downarrow_1\rangle \\ &= \frac{1}{2}(|\uparrow_2\rangle\langle\uparrow_2| + |\downarrow_2\rangle\langle\downarrow_2|) \\ &= \frac{1}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}_2.\end{aligned} \tag{18.1.12}$$

Likewise, the density matrix corresponding to the second experiment is

$$\begin{aligned}\rho_2^{<}(II) &= \mathrm{Tr}_1[\rho_{12}^{<}(II)] = \langle +_1|\phi_{12}^{<}\rangle\langle\phi_{12}^{<}|+_1\rangle + \langle -_1|\phi_{12}^{<}\rangle\langle\phi_{12}^{<}|-_1\rangle \\ &= \frac{1}{2}(|-_2\rangle\langle -_2| + |+_2\rangle\langle +_2|),\end{aligned} \tag{18.1.13}$$

which in terms of $|\uparrow_2\rangle$ and $|\downarrow_1\rangle$ spinors becomes

$$\begin{aligned}\rho_2^{<}(II) &= \frac{1}{4}[(|\uparrow_2\rangle - |\downarrow_2\rangle)(\langle\uparrow_2| - \langle\downarrow_2|) + (|\uparrow_2\rangle + |\downarrow_2\rangle) \\ &\qquad (\langle\uparrow_2| + \langle\downarrow_2|)] \\ &= \frac{1}{2}(|\uparrow_2\rangle\langle\uparrow_2| + |\downarrow_2\rangle\langle\downarrow_2|) = \frac{1}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}_2.\end{aligned} \tag{18.1.14}$$

Hence we now have

$$\rho_2^{<}(I) = \rho_2^{<}(II), \tag{18.1.15}$$

which equality demonstrates the internal consistency of quantum theory. That is, we do not have two different descriptions of the same particle if we use the proper density matrix approach.

But even though the internal workings of quantum mechanics can be made self-consistent,* quantum mechanics seems strange. Furthermore, it is just this clever use of 'entangled' (e.g., spin singlet) states as introduced by EPR and expanded by Bell that teaches us just how strange the quantum world really is.

The inability of quantum mechanics to make definite predictions for the outcome of certain measurements led EPR to postulate the existence of 'hidden' variables which are not known and perhaps not measurable. It was hoped that an inclusion of these hidden variables would restore the completeness and determinism to the quantum theory. Bell's inequalities, to which we turn next, provide a basis for a quantitative test of the hidden variable approach. It is shown that, by performing correlation experiments of the type considered in EPR's argument, one can distinguish between the predictions of certain hidden variable theories and quantum mechanics.

* See also the treatment of Griffiths and of Gell-Mann and Hartle on a consistent interpretation of quantum mechanics via quantum trajectories as discussed in *Phys. Rev. Lett.* 70, 2201 (1993) and references therein.

18.2 Bell's inequality

We consider the EPR gedanken experiment illustrated in Fig. 18.2. A spin-zero system 'splits' into two spin-1/2 particles which then have anticorrelated values of spin projection along any given axis. For the purpose of proving Bell's theorem we are interested in the probability that particle 1 will pass through a Stern–Gerlach apparatus (SGA_1) in Fig. 18.2 which is oriented at an angle θ_a with the vertical ($+z$) direction and that particle 2 will pass through a Stern–Gerlach apparatus (SGA_2) which is oriented at an angle θ_b to the vertical. We denote this joint passage probability by $P(\theta_a, \theta_b) \equiv P_{ab}$. To proceed with the proof, we first establish our notation by considering the expression,

$$P_{ab} = P\,(\overset{\text{particle 1}}{\underset{a\;\;b\;\;c}{+\;-\;\bigcirc}}\;|\;\overset{\text{particle 2}}{\underset{a\;\;b\;\;c}{-\;+\;\bigcirc}}). \tag{18.2.1}$$

Here, the left side of the partition in the expanded notation refers to particle 1 and the right side to particle 2. As shown in Eq. (18.2.1) there are three 'slots' on each side of the partition in which we have put either a plus sign, a minus sign, or a circle. The first, second, and third slots are reserved for information concerning passage through an SGA oriented at the angles θ_a, θ_b, and θ_c, respectively. A plus sign refers to passage and a minus sign to blockage. A circle means that the particular joint probability in question does not contain information about passage at that angle. So for example in Eq. (18.2.1) the first + means that particle 1 passes the SGA oriented at θ_a but then particle 2 would not pass through a SGA oriented at θ_a and this we denote by a −. Likewise if particle 2 passes through a SGA at θ_b we put a + in the record slot to the right of the vertical bar and therefore a − in the record slot associated with particle 1.

Now that we have explained the notation in general, let us return to Eq. (18.2.1). Recall that P_{ab} denotes the probability that particle 1 passes SGA_1 oriented at the angle θ_a to the z-axis and particle 2 passes SGA_2 oriented at the angle θ_b to the vertical. Likewise we write,

$$P_{bc} = P(\bigcirc\; +\; -\; |\; \bigcirc\; -\; +), \tag{18.2.2}$$

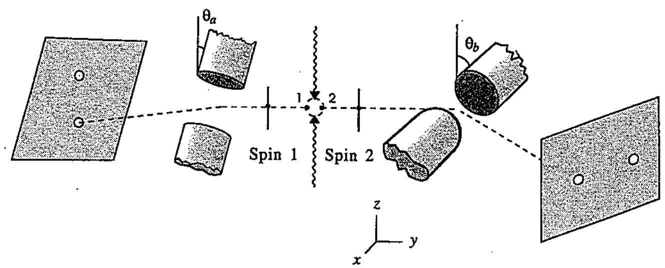

Fig. 18.2 Schematic of the EPR gedanken experiment. A spin-zero system such as orthohydrogen is split by an external field. The two spin-1/2 particles (protons) proceed in the opposite directions where they pass through the Stern-Gerlach apparati oriented at an angle θ_a with the vertical (+z) direction in the case of particle 1 and at an angle θ_b in the case of particle 2. For example the spin-up particle as indicated will 'pass' through the lower hole.

and

$$P_{ac} = P(+ \bigcirc - \mid - \bigcirc +). \tag{18.2.3}$$

The usefulness of this notation becomes apparent when we take the next step. Although the joint probability P_{ab} says nothing about passage at θ_c we do know that for any given particle the probability that it will pass an SGA oriented at θ_c to the vertical plus the probability that it will not pass such an apparatus must be equal to unity. Using this fact and the anticorrelation of the spin projections we write,

$$\begin{aligned} P_{ab} &= P(+ - \bigcirc \mid - + \bigcirc) \\ &= P(+ - + \mid - + -) + P(+ - - \mid - + +). \end{aligned} \tag{18.2.4}$$

Similarly,

$$\begin{aligned} P_{bc} &= P(\bigcirc + - \mid \bigcirc - +) \\ &= P(+ + - \mid - - +) + P(- + - \mid + - +), \end{aligned} \tag{18.2.5}$$

$$\begin{aligned} P_{ac} &= P(+ \bigcirc - \mid - \bigcirc +) \\ &= P(+ + - \mid - - +) + P(+ - - \mid - + +). \end{aligned} \tag{18.2.6}$$

Given Eqs. (18.2.4)–(18.2.6), the proof of Bell's theorem easily follows. We add P_{ab} and P_{bc} to get,

$$\begin{aligned} P_{ab} + P_{bc} = &P(+ - + \mid - + -) + P(+ - - \mid - + +) \\ &+ P(+ + - \mid - - +) + P(- + - \mid + - +). \end{aligned} \tag{18.2.7}$$

We note that, using Eq. (18.2.6), Eq. (18.2.7) can be written as,

$$\begin{aligned} P_{ab} + P_{bc} = &P_{ac} + P(+ - + \mid - + -) \\ &+ P(- + - \mid + - +). \end{aligned} \tag{18.2.8}$$

Classically, probabilities must be positive so that this implies,

$$P_{ab} + P_{bc} \geq P_{ac}. \tag{18.2.9}$$

This completes our proof of Bell's theorem.

18.3 Quantum calculation of the correlations in Bell's theorem

The quantum calculation for the probability of a spin-1/2 particle described by a state vector $|\Psi\rangle$ passing through a SGA oriented at angle θ is given by

$$P_\Psi(\theta) = |\langle\theta|\Psi\rangle|^2, \tag{18.3.1}$$

where the state $|\theta\rangle$ is formed by rotating a 'spin up' state about the y-axis

$$|\theta\rangle = e^{-i\theta\sigma_y/2}|\uparrow\rangle. \tag{18.3.2}$$

Here we recall that

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{18.3.3}$$

and

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{18.3.4}$$

We may rewrite (18.3.1) as

$$P_\Psi(\theta) = \langle\Psi|\theta\rangle\langle\theta|\Psi\rangle. \tag{18.3.5}$$

Now the projection operator $|\theta\rangle\langle\theta|$ is a useful quantity which we define as

$$\pi_\theta = |\theta\rangle\langle\theta|. \tag{18.3.6}$$

From Eq. (18.3.2) this may be written as

$$\pi_\theta = e^{-i\theta\sigma_y/2}|\uparrow\rangle\langle\uparrow| e^{i\theta\sigma_y/2}, \tag{18.3.7}$$

and using the fact that

$$e^{-i\theta\sigma_y/2}|\uparrow\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + \sin\frac{\theta}{2}|\downarrow\rangle, \tag{18.3.8}$$

we find that Eq. (18.3.7) becomes

$$\pi_\theta = \frac{1}{2}(1 + \sigma_z \cos\theta + \sigma_x \sin\theta). \tag{18.3.9}$$

Now from the previous discussion we see that the probability of simultaneous passage through SGAs at θ_a and θ_b by the particles described by the spin singlet state Eq. (18.1.1) is

$$P_{ab} = \langle \Psi_{1,2} | \pi^{(1)}_{\theta_a} \pi^{(2)}_{\theta_b} | \Psi_{1,2} \rangle, \tag{18.3.10}$$

where the projection operators $\pi^{(1)}_{\theta_a}$ and $\pi^{(2)}_{\theta_b}$ correspond to particles 1 and 2. After a little algebra, see Problem 18.3, we find

$$P_{ab} = \frac{1}{4}[1 - \cos(\theta_a - \theta_b)] = \frac{1}{2}\sin^2\left(\frac{\theta_a - \theta_b}{2}\right). \tag{18.3.11}$$

Now in our derivation of Bell's inequality (18.2.9) we considered only three angles. Hence, we may use our quantum mechanical result

$$P_{ab} = \frac{1}{2}\sin^2\left(\frac{\theta_a - \theta_b}{2}\right), \tag{18.3.12}$$

to check whether Bell's theorem is 'obeyed' by quantum mechanics. That is, is the 'quantum version' of Bell's inequality obeyed?

$$\frac{1}{2}\sin^2\left(\frac{\theta_a - \theta_b}{2}\right) + \frac{1}{2}\sin^2\left(\frac{\theta_b - \theta_c}{2}\right) \geq \frac{1}{2}\sin^2\left(\frac{\theta_a - \theta_c}{2}\right). \tag{18.3.13}$$

To answer this we need only consider the angles $\theta_a = 0$, $\theta_b = \pi/4$, and $\theta_c = \pi/2$, so that Eq. (18.3.13) implies

$$2\sin^2\frac{\pi}{8} \geq \sin^2\frac{\pi}{4},$$

or

$$0.15 \geq 0.25, \tag{18.3.14}$$

which is false and therefore quantum mechanics violates the Bell's inequality!

Thus we have a clear situation in which the predictions of quantum mechanics and hidden variable theory are at variance. Many experiments have been, and continue to be, carried out and all of the experiments to date favor quantum mechanics, as seen below. There are still a few 'loopholes' which leave the question open but most workers now believe that the ultimate experiment* will support quantum mechanics.

What is it that went wrong in our derivation of Bell's theorem? How could such simple arguments be wrong? Perhaps the best way to answer such question is through the study of the simple examples that are considered in the next section.

* Kwiat, Eberhard, Steinberg, and Chiao [1994] and Fry, Walther, and Li [1995].

It may be noted that there are a number of other Bell inequalities. One useful form of the Bell inequalities (which is usually tested in experiments) is due to Clauser and Horne, and is given by (see Problem 18.1)

$$S \leq 1, \tag{18.3.15}$$

where

$$S = \frac{P_{12}(\theta_a, \theta_b) - P_{12}(\theta_a, \theta'_b) + P_{12}(\theta'_a, \theta_b) + P_{12}(\theta'_a, \theta'_b)}{P_1(\theta'_a) + P_2(\theta_b)}, \tag{18.3.16}$$

with $P_1(\theta'_a)$ and $P_2(\theta_b)$ being the passage probabilities for particles 1 and 2 to pass through the respective Stern–Gerlach apparati at angles θ'_a and θ_b, respectively.

In many experiments to test the Bell's inequalities, certain symmetries help to simplify the inequality (18.3.15). In these experiments $P_1(\theta'_a)$ and $P_2(\theta_b)$ are independent of the angles θ'_a and θ_b respectively, i.e., $P_1(\theta'_a) \equiv P_1$ and $P_2(\theta_b) \equiv P_2$. In addition the joint probabilities $P_{12}(\theta_a, \theta_b)$ depend only on the magnitude of the difference of the angles θ_a and θ_b, i.e., $P_{12}(\theta_a, \theta_b) \equiv P_{12}(|\theta_a - \theta_b|)$. Suppose that we chose θ_a, θ_b, θ'_b, and θ'_a in (18.3.16) so that

$$|\theta_a - \theta_b| = |\theta'_a - \theta_b| = |\theta'_a - \theta'_b| = \frac{1}{3}|\theta_a - \theta'_b| = \alpha. \tag{18.3.17}$$

We then have

$$S(\alpha) = \frac{3P_{12}(\alpha) - P_{12}(3\alpha)}{P_1 + P_2}. \tag{18.3.18}$$

Most experiments have been a variation of an experiment in which one measures the polarization correlations of the photons emitted successively in an atomic cascade. In such experiments, a three-level atom proceeds from, for example, a $J = 0$ level to a $J = 1$ level, and terminates in a $J = 0$ level which is the atomic ground state. Typically the atomic level scheme in calcium is employed where the $4p^2\ {}^1S_0$ level is populated by laser radiation via two-photon excitation. It then decays to the $4s^2\ {}^1S_0$ state via the $4p4s^1\ P_1$ level emitting two photons of wavelengths 5513 Å and 4227 Å (see Fig. 18.3). Due to parity and angular momentum conservation, there is a strong correlation in the polarization of the emitted photons.

The schematics of the experiment are shown in Fig. 18.4. The pair of correlated visible photons are emitted in the atomic cascade in a well-stabilized high-efficiency source S. These photons pass through the switching devices C_1 and C_2, followed by two polarizers in two different orientations: θ_a and θ'_a on side 1, and θ_b and θ'_b on side 2. The

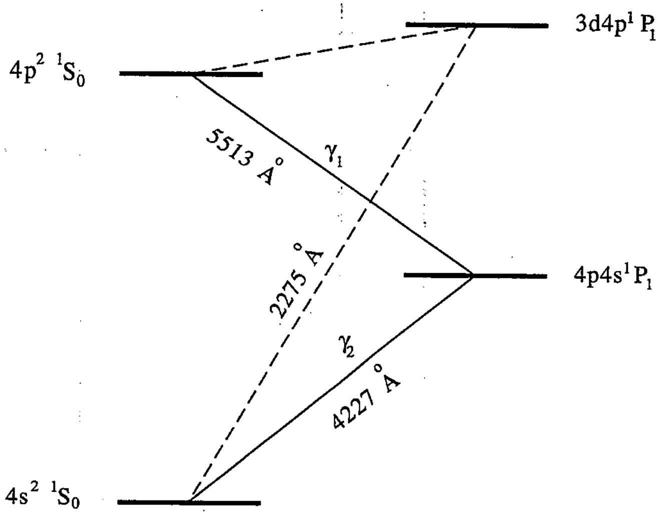

Fig. 18.3 Level scheme of calcium. The two-photon route for excitation to the upper level $4p^2\,{}^1S_0$ is shown by dashed lines.

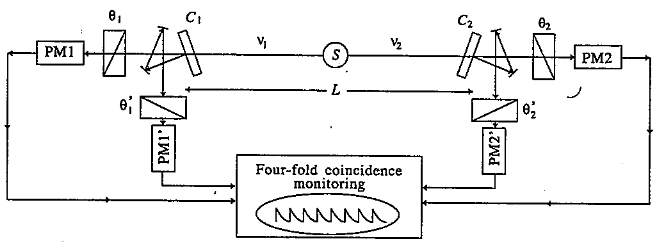

Fig. 18.4 Schematic of the photon correlation experiment in two-photon cascade emission to test Bell's inequality. (From A. Aspect, J. Dalibard, and G. Roger, *Phys. Rev. Lett.* **49**, 1804 (1982).)

photon multipliers PM1, PM2, PM1', and PM2' and the coincidence counting electronics measures the joint probabilities.

The two photons are distinguishable by their wavelengths or frequencies. We assume the emitting atom to be at the origin and consider the emitted photons which counter-propagate in the $\pm y$-directions. An optical filter in the $+y$-direction transmits only photons of frequency ν_1 and a filter in the $-y$-direction transmits only photons of frequency ν_2. As the transition is $J = 0 \rightarrow J = 1 \rightarrow J = 0$, the initial and final states of the atom have zero angular momentum and the same parity. Similarly, the two-photon state must have zero angular momentum and even parity. The state of the polarization of the two photons, after the passage through the filters, is of the form

$$|\Psi_{1,2}\rangle = \frac{1}{\sqrt{2}}(|R_1\rangle|R_2\rangle + |L_1\rangle|L_2\rangle), \tag{18.3.19}$$

where R and L refer to the photon polarizations being right and left circular and the subscripts 1 and 2 refer to the photons having frequencies ν_1 and ν_2, respectively. A change of basis to linear polarization states $|x\rangle$, $|z\rangle$ allows the state vector (18.3.19) to be rewritten as

$$|\Psi_{1,2}\rangle = \frac{1}{\sqrt{2}}(|z_1\rangle|z_2\rangle + |x_1\rangle|x_2\rangle). \tag{18.3.20}$$

The joint linear polarization measurement made by polarizers at angles θ_a and θ_b to the z-axis projects the state of Eq. (18.3.20) onto the two polarization states

$$|\theta_a\rangle = \cos\theta_a|z_1\rangle + \sin\theta_a|x_1\rangle, \tag{18.3.21}$$
$$|\theta_b\rangle = \cos\theta_b|z_2\rangle + \sin\theta_b|x_2\rangle. \tag{18.3.22}$$

The quantum mechanical probability for passage through the two polarization analyzers is therefore given by

$$\begin{aligned} P_{12}(\theta_a, \theta_b) &= |\langle\theta_a|\langle\theta_b|\Psi_{1,2}\rangle|^2, \\ &= \frac{1}{2}\cos^2(\theta_a - \theta_b). \end{aligned} \tag{18.3.23}$$

Next we calculate $P_1(\theta)$ and $P_2(\theta)$. If the incident photon of frequency ν_1 is polarized along the x-axis, then the probability of passing the polarizer oriented at an angle θ with the x-axis, with

$$|\Psi_1\rangle = \cos\theta|x\rangle + \sin\theta|z\rangle, \tag{18.3.24}$$

is $\cos^2\theta$. However, as the incident beam is unpolarized, we average over all values of θ, i.e.,

$$\begin{aligned} P_1(\theta) &= \frac{1}{2\pi}\int_0^{2\pi} \cos^2\theta d\theta \\ &= \frac{1}{2}. \end{aligned} \tag{18.3.25}$$

Similarly

$$P_2(\theta) = \frac{1}{2}. \tag{18.3.26}$$

We now substitute the values of $P_{12}(\theta_a, \theta_b)$, $P_1(\theta)$, and $P_2(\theta)$ from Eqs. (18.3.23), (18.3.25), and (18.3.26), respectively, in Eq. (18.3.18),

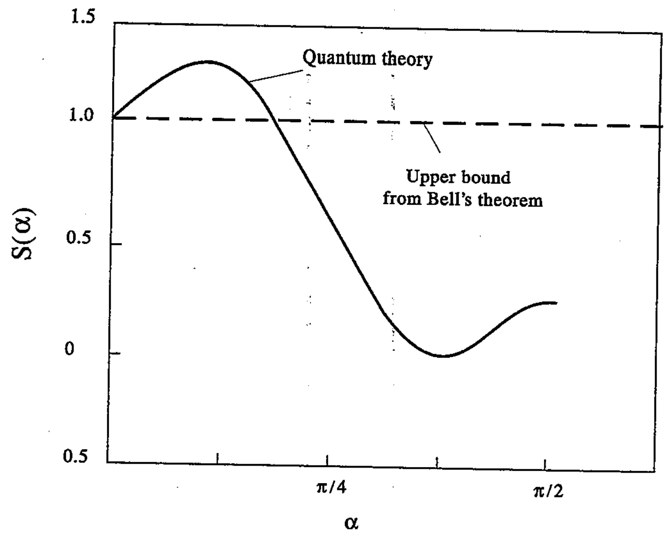

Fig. 18.5 $S(\alpha)$ versus α as given by Eq. (18.3.27). The dashed line shows the upper bound from Bell's inequality. (From J. F. Clauser and A. Shimony, *Rep. Prog. Phys.* 41, 1881 (1978).)

and obtain

$$S(\alpha) = \frac{3P_{12}(\alpha) - P_{12}(3\alpha)}{P_1 + P_2}$$
$$= \frac{1}{2}(3\cos^2\alpha - \cos^2 3\alpha). \tag{18.3.27}$$

For $\alpha = 22.5°$ this reduces to

$$S(22.5°) = 1.207, \tag{18.3.28}$$

in clear violation of the Bell's inequality (18.3.15). In Fig. 18.5, $S(\alpha)$ is plotted against α. It is seen that Bell's inequality (18.3.15) is violated for $0 < \alpha < 3\pi/16$.

18.4 Hidden variables from a quantum optical perspective*

Belinfante in his scholarly book on hidden variable (HV) theory shows that HV theories are not so far from quantum mechanics (QM) as might be thought. Stimulated by Belinfante's treatment and observations, one is led to apply quantum distribution theory to the

* This section follows Scully [1983]; for further reading see Mermin [1993].

spin-1/2 system, much as we want to do in quantum optics. In so doing a hidden variable theory is suggested which is in agreement with quantum theory insofar as the two-particle correlation experiments are concerned, but is clearly nonlocal.

Let us begin by recalling the joint probability that particle 1 is passed through a SGA oriented at an angle θ_a to the vertical ($+z$) direction and that particle 2 is passed through a SGA oriented at an angle θ_b to the vertical, as given by the correlation function

$$P_{12}(\theta_a, \theta_b) = \langle\Psi|\pi^{(1)}_{\theta_a}\pi^{(2)}_{\theta_b}|\Psi\rangle, \tag{18.4.1}$$

and for the spin singlet we found (see Eq. (18.3.11))

$$P_{12}(\theta_a, \theta_b) = \frac{1}{4}[1 - \cos(\theta_a - \theta_b)]. \quad \text{(QM)} \tag{18.4.2}$$

Next we consider the same problem following Belinfante; we require that in order to give hidden variable theories an air of possibility we want them to yield the same results as quantum mechanics, at least in the simplest cases. For example, in an unpolarized beam only 1/2 the particles should pass through a given SGA. Further, the probability of passing through a second SGA placed behind (and at an angle θ) relative to the previous (vertical) SGA should be given by $\langle\uparrow|\pi_\theta|\uparrow\rangle$ which is $(1 + \cos\theta)/2$. Or, more generally, if a spin emerges from a SGA oriented at an angle α and then passes into a SGA tipped through an angle θ relative to the vertical, then the likelihood that the particle will emerge from the second SGA is given by

$$\frac{1}{2}[1 + \cos(\theta - \alpha)]. \tag{18.4.3}$$

Thus we might say that a 'hidden variable' α determined whether the spin passed through the apparatus whose angle θ is determined by the experimenter.

With this in mind we define the hidden variable probability function

$$\tilde{\pi}_\theta(\alpha) \equiv \frac{1}{2}[1 + \cos(\theta - \alpha)], \tag{18.4.4}$$

as giving the probability of 'simultaneous passage' through the SGAs oriented at θ and α.

We proceed now to consider the case where the two spins of our singlet system of Fig. 18.2 (having polarization angles α and β for spins 1 and 2) are correlated such that

$$I(\alpha, \beta)d\alpha d\beta \tag{18.4.5}$$

Now, following the same approach as before, and letting α represent the orientation of $\mathbf{m}^{(1)}$, and β that of $\mathbf{m}^{(2)}$, with ϕ the arbitrary orientation of the z-axis, this may be rewritten as

$$\begin{aligned}
&P(\mathbf{m}^{(1)}, \mathbf{m}^{(2)}) \\
&= \frac{1}{4}\Big[\ \delta\left(\alpha-\phi-\frac{3\pi}{4}\right)\delta\left(\beta-\phi+\frac{\pi}{4}\right) \\
&\quad + \delta\left(\alpha-\phi-\frac{\pi}{4}\right)\delta\left(\beta-\phi+\frac{3\pi}{4}\right) \\
&\quad + \delta\left(\alpha-\phi+\frac{3\pi}{4}\right)\delta\left(\beta-\phi-\frac{\pi}{4}\right) \\
&\quad + \delta\left(\alpha-\phi+\frac{\pi}{4}\right)\delta\left(\beta-\phi-\frac{3\pi}{4}\right)\Big] \\
&= \frac{1}{4}\delta(\alpha-\beta-\pi)\Big[\delta\left(\alpha-\phi-\frac{3\pi}{4}\right)+\delta\left(\alpha-\phi-\frac{\pi}{4}\right) \\
&\qquad + \delta\left(\alpha-\phi+\frac{3\pi}{4}\right)+\delta\left(\alpha-\phi+\frac{\pi}{4}\right)\Big],
\end{aligned} \tag{18.B.7}$$

which is Eq. (18.4.20).

Problems

18.1 (a) Consider four numbers x_1, x_2, x_3, and x_4 such that $0 \le x_i < 1$, $(i = 1, 2, 3, 4)$. Show that the function

$$X = x_1x_2 - x_1x_4 + x_2x_3 + x_3x_4 - x_2 - x_3,$$

is constrained by the inequality

$$-1 \le X \le 0.$$

(b) If we choose $x_1 = P_1(\mu, \theta_a)$, $x_2 = P_2(\mu, \theta_b)$, $x_3 = P_1(\mu, \theta'_a)$, and $x_4 = P_2(\mu, \theta'_b)$ in the above inequality, then prove the following Bell's inequality:

$$S \le 1,$$

where

$$S = \frac{P_{12}(\theta_a,\theta_b)-P_{12}(\theta_a,\theta'_b)+P_{12}(\theta'_a,\theta_b)+P_{12}(\theta'_a,\theta'_b)}{P_1(\theta'_a)+P_2(\theta_b)}$$

with

$$P_1(\theta_a) = \int d\Lambda P_1(\mu, \theta_a),$$

$$P_2(\theta_b) = \int d\Lambda P_2(\mu, \theta_b),$$

$$P_{12}(\theta_a, \theta_b) = \int d\Lambda P_1(\mu, \theta_a) P_2(\mu, \theta_b).$$

Here $P_1(\mu, \theta_a)$ and $P_2(\mu, \theta_b)$ are the probabilities of detecting particles 1 and 2 with the orientation of Stern–Gerlach apparati in Fig. 18.2 at angles θ_a and θ_b, respectively, where μ are the hidden variables that describe 'completely' the emission process in the source, and $d\Lambda$ is a measure of the variables μ. Now $P_1(\theta'_a)$ and $P_2(\theta_b)$ are the passage probabilities for particles 1 and 2 to pass through the respective Stern–Gerlach apparati oriented at angles θ'_a and θ_b, respectively, and $P_{12}(\theta_a, \theta_b)$ is the joint probability that particles 1 and 2 will pass through their respective Stern–Gerlach apparati oriented at angles θ_a and θ_b, respectively. (Hint: See J. F. Clauser and M. A. Horne, *Phys. Rev. D* **10**, 526 (1974).)

18.2 (a) Show that

$$e^{-i\theta\sigma_y/2} = \cos\left(\frac{\theta}{2}\right) - i\sigma_y \sin\left(\frac{\theta}{2}\right).$$

(b) Use this result to show that

$$\begin{aligned}
&e^{-i\theta\sigma_y/2}|\uparrow\rangle\langle\uparrow|e^{i\theta\sigma_y/2} \\
&= \frac{1}{2}(1 + \sigma_z\cos\theta + \sigma_x\sin\theta).
\end{aligned}$$

18.3 For a spin singlet state

$$|\Psi_{1,2}\rangle = \frac{1}{\sqrt{2}}[|\uparrow_1, \downarrow_2\rangle - |\downarrow_1, \uparrow_2\rangle],$$

show that

$$\langle\Psi_{1,2}|\pi_{\theta_a}\pi_{\theta_b}|\Psi_{1,2}\rangle = \frac{1}{4}[1 - \cos(\theta_a^1 - \theta_b^2)],$$

where

$$\pi^{(1)}_{\theta_a} = \frac{1}{2}[1 + \sigma^{(1)}_z \cos\theta_a + \sigma^{(1)}_x \sin\theta_a],$$
$$\pi^{(2)}_{\theta_b} = \frac{1}{2}[1 + \sigma^{(2)}_z \cos\theta_b + \sigma^{(2)}_x \sin\theta_b].$$

Here $\sigma^{(i)}_x$ and $\sigma^{(i)}_z$ are the Pauli matrices for the ith spin.

18.4 Show that

$$|\Psi\rangle_3 = \frac{1}{\sqrt{2}}(|\uparrow_1\uparrow_2\uparrow_3\rangle - |\downarrow_1\downarrow_2\downarrow_3\rangle)$$

is an eigenstate of the operators $\sigma^{(1)}_y\sigma^{(2)}_x\sigma^{(3)}_y$ and $\sigma^{(1)}_y\sigma^{(2)}_y\sigma^{(3)}_x$.

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where

$$\psi \equiv \theta_1 - \theta_2 \, .$$

If we choose

$$\psi = \theta_2 - \theta_1 = \theta_1' - \theta_2 = \theta_1' - \theta_2' = \frac{1}{3}(\theta_1 - \theta_2') \, ,$$

one finds

$$B = 3\cos 2\psi - \cos 6\psi \, . \tag{13.27}$$

When $\psi = 22.5°$, $B = 2\sqrt{2}$ showing a clear violation of the Bell inequality $|B| \leq 2$.

This violation has convincingly been demonstrated in the experiment of Aspect [4]. In this experiment the polarisation analysers were essentially beam splitters with polarisation-dependent transmittivity. Ideally, one would like to have the transmittivity (T^+) for the modes a_+ and b_+ equal to one, and the reflectivity (R^-) for the modes a_- and b_- also equal to one. However, in the experiment the measured values were $T_1^+ = R_1^- = 0.950$, $T_1^- = R_1^+ = 0.007$ and $T_2^+ = T_2^- = 0.930$, $T_2^- = R_2^+ = 0.007$.

The expression for $E(\theta_1, \theta_2)$ is then modified:

$$E(\theta_1, \theta_2) = F\frac{(T_1^+ - T_1^-)(T_2^+ - T_2^-)}{(T_1^+ + T_1^-)(T_2^+ + T_2^-)}\cos 2\psi \, , \tag{13.28}$$

where F is a geometrical factor accounting for finite solid angles of detection. In this experiment $F = 0.984$, and quantum mechanics would give for $\psi = 22.5°$, $B = 2.7$.

The observed value was 2.697 ± 0.015, in quite good agreement with quantum theory and a clear violation of the Bell inequality. In Fig. 13.2 is shown a plot of the theoretical and experimental results as a function of ψ. The agreement with quantum mechanics is better than 1%. It would appear in the light of this experiment that realistic local theories for completing quantum mechanics are untenable.

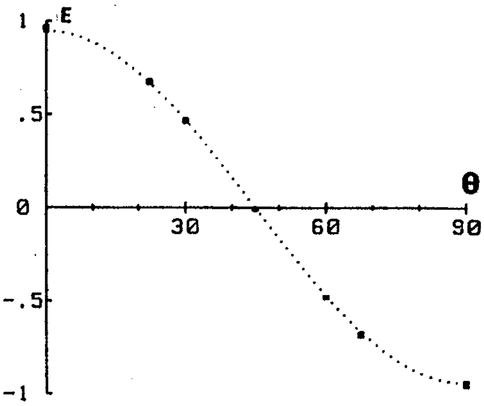

Fig. 13.2 Correlation of polarisations as a function of the relative angle of the polarisation analysers. The indicated errors are ±2 standard deviations. The *dotted curve* is the quantum-mechanical prediction for the experiment. For ideal polarisers the curves woudl reach the values ±1. (From Aspect et al. Phys. Rev. Letts. **49**, 92 (1982))