Umeå University
Department of Physics

Excercises

Quantum optics

Växelverkan mellan ljus och materia

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Preface

Preface 2005

This collection of problems and exercises has been compiled specifically for the course ‘Quantum optics’ (‘Växelverkan mellan ljus och materia’). Many of the included problems have been copied from the course textbook ‘The Quantum Theory of Light’ by Rodney Loudon. We have put them here in order to have one unified collection of problems and also because some of them have been slightly altered in order to fit our needs. A fair number of the problems are taken from previous exams and previous assignments. This also means that this collection may well grow with time.

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Chapter 1

Phenomenological approach to quantum optics

1:1. Prove that the maximum value $\langle W_T(\omega)\rangle_{\max}$ of the energy density and the frequency $\omega_{\max}$ at which it occurs are related by

$$\langle W_T(\omega)\rangle_{\max} = \frac{\omega_{\max}^2}{\pi^2 c^3}\left(3k_{\mathrm{B}}T - \hbar\omega_{\max}\right)$$

Show by numerical trial and error or by drawing a rough graph that the value of the frequency is given approximately by $\omega_{\max} = 2.8k_{\mathrm{B}}T/\hbar$. This is known as Wien's displacement law.

1:2. The rate equation for a two-level atom in a field is:

$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = -\frac{\mathrm{d}N_2}{\mathrm{d}t} = N_2 A + (N_2 - N_1)B\langle W\rangle$$

Prove that the general solution of this is.

$$N_1(t) = \left\{N_1(0) - N\frac{W_{\mathrm{s}} + \langle W\rangle}{W_{\mathrm{s}} + 2\langle W\rangle}\right\}\exp\left[-(A + 2B\langle W\rangle)t\right] + N\frac{W_{\mathrm{s}} + \langle W\rangle}{W_{\mathrm{s}} + 2\langle W\rangle}$$

The solution for $N_2(t)$ follows from $N = N_1 + N_2$.

1:3. Derive expressions for the rates of change of the atomic excitation energy as functions of time for

(a) an atom initially in its groundstate, illuminated by a beam of energy density $\langle W\rangle$,

(b) an atom initially in its continously illuminated state, with the beam switched off at time $t = 0$.

1:4. A light beam that illuminates atoms is repeatedly turned on and off, with the same duration τ for the on and off periods. The mean number of excited atoms settles into a regular pattern after many on/off cycles have taken place. Sketch the expected form of this regular variation and show that the maximum number of excited atoms is

$$N_2(t) = N\frac{\langle W\rangle}{W_s + 2\langle W\rangle}\frac{1-\exp\left[-(A+2B\langle W\rangle)\tau\right]}{1-\exp\left[-2(A+B\langle W\rangle)\tau\right]}.$$

Investigate the limiting forms when $\tau \to \infty$ and $\tau \to 0$ and explain them in physical terms.

1:5. (a) An atom with transition frequency $\omega = 3\times10^{15}\,\mathrm{s}^{-1}$ and radiative lifetime $10^{-7}\,\mathrm{s}$ is illuminated by a light beam whose energy density equals the saturation value. What fraction of the time does the atom on average spend in its excited state in steady-state conditions?

What are the average numbers per second of:

(b) absorptions,

(c) spontaneous emissions and

(d) stimulated emissions?

1:6. Figure 1.1 shows the rates of absorption, stimulated emission and spontaneous emission as a function of the mean value of the energy density of the radiation. The curves are derived using only the phenomenological Einstein theory and rate equations for a two-level system. No quantum mechanics have been used. The energy density is given in units of the saturation energy density.

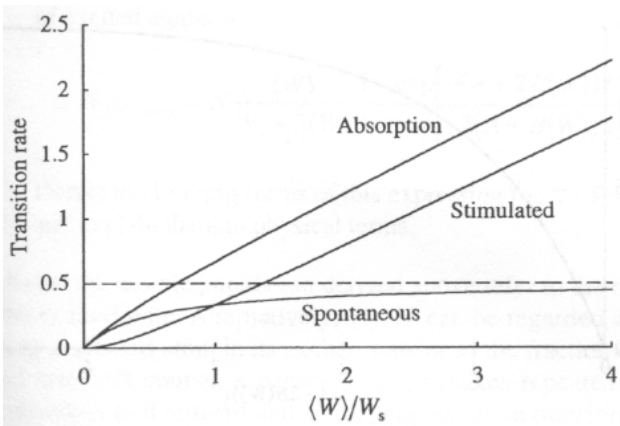


Figure 1.1: Mean rates of the three Einstein transitions in units of the A coefficient as functions of the radiative energy density.

Make an interpretation of the curves. What does it mean that they look like they do? How will the corresponding populations in the upper and the lower state depend on the energy density? How will an extra applied beam of resonant light be affected if it interacts with this system, under the conditions of high and low energy density respectively?

Chapter 2

Semi-classical description of light-atom interaction

2:1. Consider a two-level atom where

$$\Psi(\mathbf{r},t) = C_1\Psi_1(\mathbf{r},t) + C_2\Psi_2(\mathbf{r},t).$$

The coupled equations of motion for the coefficients C_1 and C_2, when the atom interacts with a field are:

$$\begin{aligned} C_2\Omega\cos(\omega t)\exp(-\mathrm{i}\omega_0 t) &= \mathrm{i}\frac{\mathrm{d}C_1}{\mathrm{d}t} \\ C_1\Omega\cos(\omega t)\exp(-\mathrm{i}\omega_0 t) &= \mathrm{i}\frac{\mathrm{d}C_2}{\mathrm{d}t} \end{aligned}$$

where Ω is the Rabi frequency.

Prove that $|C_1|^2 + |C_2|^2$ does not change with time, thus ensuring that the normalization condition remains valid.

2:2. The optical Bloch equations can be written as:

$$\begin{aligned} \frac{\mathrm{d}\tilde{\rho}_{22}}{\mathrm{d}t} &= -\frac{\mathrm{d}\tilde{\rho}_{22}}{\mathrm{d}t} = & -\frac{\mathrm{i}}{2}\Omega(\tilde{\rho}_{12} - \tilde{\rho}_{21}) \\ \frac{\mathrm{d}\tilde{\rho_{12}}}{\mathrm{d}t} &= \frac{\mathrm{d}\tilde{\rho}_{21}^*}{\mathrm{d}t} = & \frac{\mathrm{i}}{2}\mathrm{i}\Omega(\tilde{\rho}_{11} - \tilde{\rho}_{22}) + \mathrm{i}(\omega_0 - \omega)\tilde{\rho}_{12} \end{aligned}$$

Show that for the initial conditions $\tilde{\rho}_{22}(0) = 0$ and $\tilde{\rho}_{12}(0) = 0$ the solution is $\tilde{\rho}_{22}(t) = \left(\frac{\Omega}{\Omega_{\mathrm{gen}}}\right)\sin^2\left(\frac{\Omega_{\mathrm{gen}}t}{2}\right)$.

2:3. Prove that the Doppler and collisional contributions to the linewidth of an atomic transition are equal at a gas density for which the volume per atom is close to λd^2, where λ is the optical wavelength of the transition and d is the distance between the centers of the atoms during a collision.

2:4. Consider a three-level atom, where the state can be described by:

$$\Psi(\mathbf{r},t) = C_1\Psi_1(\mathbf{r},t) + C_2\Psi_2(\mathbf{r},t) + C_3\Psi_3(\mathbf{r},t)$$

Assume that state Ψ_1 and Ψ_2 have odd parity, whereas Ψ_3 has even parity, and all relevant resonant radiation frequencies are present. Formulate the coupled equations of motion for the coefficients C_1, C_2 and C_3.

2:5. Assume a two level atom, where a resonant radiation field is suddenly turned on. Show how an expression for the degree of excitation in the upper level as a function of time can be derived from the optical Bloch equations, when radiative damping is included in the model. The rotating-wave approximation and the dipole approximation are allowed.

When solving this problem (and other problems involving the density matrix for a two state system) it is convinient to introduce

$$\begin{aligned}
\mathcal{X} &= \frac{1}{2}\left(\rho_{11}-\rho_{22}\right)\\
\mathcal{Y} &= -\frac{1}{2}\mathrm{i}\left(\rho_{12}-\rho_{21}\right) = -\frac{1}{2}\mathrm{i}\left(\rho_{12}-\rho_{12}^{*}\right)\\
\mathcal{Z} &= \frac{1}{2}\left(\rho_{12}+\rho_{21}\right) = \frac{1}{2}\left(\rho_{12}+\rho_{12}^{*}\right)
\end{aligned}$$

which can be inverted as (we also use that $\rho_{11}+\rho_{22}=1$)

$$\begin{aligned}
\rho_{11} &= \mathcal{X}+\frac{1}{2}\\
\rho_{22} &= \frac{1}{2}-\mathcal{X}\\
\rho_{12} &= \mathcal{Z}+\mathrm{i}\mathcal{Y}\\
\rho_{21} &= \mathcal{Z}-\mathrm{i}\mathcal{Y}
\end{aligned}$$

note that $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are real quantities.

2:6. The equation of motion for the density matrix in the absence of spontaneous emission is

$$\begin{aligned}
\frac{\mathrm{d}\rho_{22}}{\mathrm{d}t} &= -\mathrm{i}\Omega\cos(\omega t)\left\{\exp(\mathrm{i}\omega_0 t)\rho_{12}-\exp(-\mathrm{i}\omega_0 t)\rho_{21}\right\}\\
\frac{\mathrm{d}\rho_{12}}{\mathrm{d}t} &= \mathrm{i}\Omega\cos(\omega t)\exp(-\mathrm{i}\omega_0 t)(\rho_{11}-\rho 22)\\
\frac{\mathrm{d}\rho_{11}}{\mathrm{d}t} &= -\frac{\mathrm{d}\rho_{22}}{\mathrm{d}t}\\
\frac{\mathrm{d}\rho_{21}}{\mathrm{d}t} &= \frac{\mathrm{d}\rho_{12}^{*}}{\mathrm{d}t}.
\end{aligned}$$

Find the solution to these under the initial condition $\rho_{11}(t=0)=1$ and $\rho_{12}(t=0)=\rho_{21}(t=0)=\rho_{22}(t=0)=0$.

Compare your solution with the solution obtained when the rotating wave approximation is made

$$\rho_{22}(t) = \left(\frac{\Omega}{\Omega_{\mathrm{gen}}}\right)^2 \sin^2\left(\frac{1}{2}\Omega_{\mathrm{gen}}t\right)$$

where Ω_{gen} is the generalised Rabi frequency

$$\Omega_{\mathrm{gen}} = \sqrt{(\omega-\omega_0)^2+\Omega^2}$$

2:7. Consider a gas of Cs atoms, where the lifetime of a relevant excited state is 30.5 ns. Assume no other broadening mechanisms other than natural linewidth and Doppler braodening. At what temperature are the homogeneous and inhomogeneous linewidths of the same order?

2:8. In the semi-classical approximation, spontaneous emission is not inherently included in the model. The optical Bloch equations are derived from the equations of motion for the coefficients $C_j(t)$ that define the atomic wave function in the Schrödinger representation. For a two level atom, $C_1(t)$ and $C_2(t)$ are obviously defined through:

$$\Psi(\mathbf{r},t) = C_1(t)\Psi_1(\mathbf{r},t) + C_2(t)\Psi_2(\mathbf{r},t) \quad .$$

The equations of motion for $C_1(t)$ and $C_2(t)$ are

$$\mathrm{i}\frac{\mathrm{d}C_1}{\mathrm{d}t} = \Omega\cos\omega t\exp(-\mathrm{i}\omega_0 t)C_2(t)$$

$$\mathrm{i}\frac{\mathrm{d}C_2}{\mathrm{d}t} = \Omega\cos\omega t\exp(\mathrm{i}\omega_0 t)C_1(t) \quad ,$$

where Ω is the Rabi frequency, ω_0 is the atomic resonance frequency, and ω is the frequency of the light. From this, one can derive the optical Bloch equations in the rotating wave approximation:

$$\begin{aligned}
\frac{\mathrm{d}\tilde{\rho}_{22}}{\mathrm{d}t} &= -\frac{\mathrm{d}\tilde{\rho}_{11}}{\mathrm{d}t} = -\frac{\mathrm{i}}{2}\Omega(\tilde{\rho}_{12} - \tilde{\rho}_{21}) \\
\frac{\mathrm{d}\tilde{\rho}_{12}}{\mathrm{d}t} &= \frac{\mathrm{d}\tilde{\rho}_{21}^*}{\mathrm{d}t} = \frac{\mathrm{i}}{2}\Omega(\tilde{\rho}_{11} - \tilde{\rho}_{22}) + \mathrm{i}(\omega_0 - \omega)\tilde{\rho}_{12} \quad ,
\end{aligned}$$

where $\rho_{ij} = C_i C_j^*$ and

$$\begin{aligned}
\tilde{\rho}_{12} &= \exp\{\mathrm{i}(\omega_0 - \omega)t\}\rho_{12} \\
\tilde{\rho}_{21} &= \exp\{-\mathrm{i}(\omega_0 - \omega)t\}\rho_{21} \\
\tilde{\rho}_{ii} &= \rho_{ii} \quad .
\end{aligned}$$

(a) If one wants to include spontaneous emission into the semi-classical optical Bloch equations, the deacy has to be artificially added to the equations of motion. Show how this can be done and write the optical Bloch equations with this decay included (but still in the semi-classical model). The rotating wave approximation is still allowed.

(b) Sketch a graph showing how the population in the upper state varies with time (based on the model derived in a). You do not have to solve the OBE's. Just outline the general behaviour and indicate roughly a relevant time scale on the time axis. Assume that the initial state is that the atom is purely in state $|1\rangle$ and the light is off. At $t = 0$, the light turns on.

2:9. Suppose you have en ensemble of two-level atoms, with all atoms in the ground state. If you wanted to excite all atoms to an equal coherent superposition of the states $|1\rangle$ and $|2\rangle$, what would you do? The only tool at your disposal is light. Explain in sufficient detail how you would design your light field.

2:10. Assume a two level system, where the general wave function is a linear superposition:

$$\Psi(\mathbf{r},t) = C_1(t)\Psi_1(\mathbf{r},t) + C_2(t)\Psi_2(\mathbf{r},t) \quad .$$

The time-dependent wave functions Ψ_j are eigenfunctions to the atomic Hamiltonian ($\widehat{\mathcal{H}}_{\mathrm{A}}$), with eigenvalues $\mathcal{E}_j$, and ψ_j are the corresponding stationary states:

$$\Psi_j(\mathbf{r},t) = \psi_j(\mathbf{r}) \exp\left(-\mathrm{i}\mathcal{E}_j t/\hbar\right) \quad .$$

In the semi-classical picture, the light is treated as a classical electric field and the total Hamiltonian is

$$\widehat{\mathcal{H}} = \widehat{\mathcal{H}}_{\mathrm{A}} + \widehat{\mathcal{H}}_{\mathrm{I}} \quad .$$

In the interaction Hamiltonian, we only include the odd-parity electric dipole term. We also use the standard semi-classical definition of the Rabi frequency, Ω:

$$\hbar\Omega\cos\omega t = \langle 1|\widehat{\mathcal{H}}_{\mathrm{I}}|2\rangle \quad ,$$

where ω is the frequency of the light.

(a) Use the time dependent Schrödinger equation,

$$\left(\widehat{\mathcal{H}}_{\mathrm{A}} + \widehat{\mathcal{H}}_{\mathrm{I}}\right)\Psi(\mathbf{r},t) = \mathrm{i}\hbar\dot{\Psi}(\mathbf{r},t) \quad ,$$

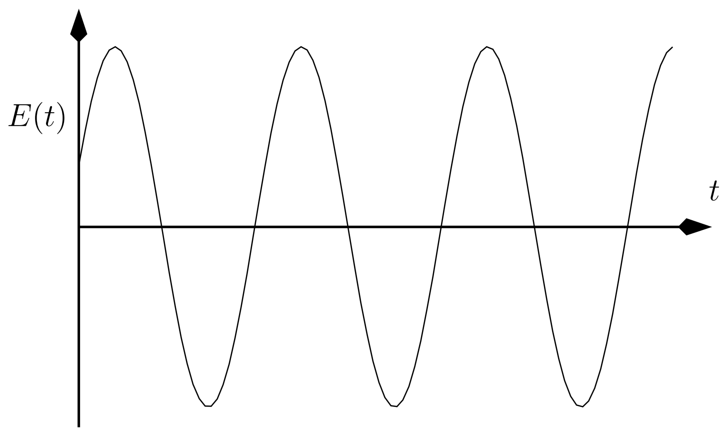
to derive the equations of motion for the coefficients $C_1(t)$ and $C_2(t)$.

(b) Rewrite these equations of motion as the optical Bloch equations, using the rotating wave approximation and the standard form for the density matrix elements $\rho_{ij} = C_i C_j^*$.

Chapter 3

Classical coherence theory

3:1 Consider a parallel light beam whose field contains a large number of contributions similar to the stable wave depicted in the figure below, all with the same frequency and wavevector but with a random distribution of phase angles. Prove that the beam is first-order coherent at any pair of space points.



3:2 Consider the beam of light produced by excitation of two stable waves, where the electric field is

$$E(z,t) = E_1 \exp(\mathrm{i}k_1 z - \mathrm{i}\omega_1 t) + E_2 \exp(\mathrm{i}k_2 z - \mathrm{i}\omega_2 t).$$

Prove that the light is first-order coherent at all pairs of points.

3:3 Consider a beam of light produced by excitation of two waves like in the problem **3:2**, but where both exhibit random amplitudes and phases. If the average intensity is equally divided between the waves, prove that

$$\left| g^{(1)}(\tau) \right| = \left| \cos\left\{ \frac{1}{2}(\omega_1 - \omega_2)\tau \right\} \right|.$$

3:4 Consider light from a source that simultaneously has collision and doppler broadening. Prove that the degree of first-order coherence is

$$g^{(1)}(\tau) = \exp\left\{ -\mathrm{i}\omega_0 \tau - \gamma_{\mathrm{coll}}|\tau| - \frac{1}{2}\Delta^2\tau^2 \right\}$$

3:5 The *Wiener-Khintchine theorem* relates the spectrum and its degree of first-order coherence through

$$F(\omega) = \frac{1}{\pi}\mathrm{Re}\int_0^\infty \mathrm{d}\tau g^{(1)}(\tau)\exp(\mathrm{i}\omega\tau)$$

(a) obtain the frequency spectrum of the excitation described in problem **3:3**

(b) Prove that the Lorentzian lineshape function

$$F_{\mathrm{L}}(\omega) = \frac{\gamma/\pi}{(\omega_0-\omega)^2+\gamma^2}$$

is correctly generated with use of the appropriate degree of first-order coherence

(c) Prove that the Gaussian lineshape function

$$F_{\mathrm{G}}(\omega) = \left(2\pi\Delta^2\right)^{-1/2}\exp\left\{-(\omega_0-\omega)^2/2\Delta^2\right\}$$

is correctly generated with use of the appropriate degree of first-order coherence

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3:6 Consider the single-mode chaotic light beam defined in problem **3:1** as a randomly phased superposition of stable waves. Prove from first principles that

$$g^{(2)}(\tau) = 2.$$

This implies that the coherence time is infinite.

3:7 Consider the light beam formed by superposition of two independent stationary beams, labelled a and b, with a total cycle-averaged intensity

$$\bar{I}(t) = \bar{I}_a(t) + \bar{I}_b(t).$$

Show that the overall degree of second-order coherence for a measurement that does not distinguish the two beams is

$$g^{(2)}(\tau) = \frac{\bar{I}_a^2 g^{(2)}_{a,a}(\tau) + \bar{I}_b^2 g^{(2)}_{b,b}(\tau) + 2\bar{I}_a\bar{I}_b}{(\bar{I}_a+\bar{I}_b)^2}.$$

Use that $g^{(2)}$ can be expressed in terms of intensities as

$$g^{(2)}(\tau) = \frac{\langle \bar{I}(t)\bar{I}(t+\tau)\rangle}{\bar{I}^2}$$

where $\bar{I}$ is the long-term average intensity.

3:8 Consider the classical electric field of a plane parallel light beam, made up of independent contributions from a large ensemble of radiating atoms:

$$E(t) = \sum_{i=1}^{\nu} E_i(t) \quad ,$$

where ν is the number of atoms. The first order electric field correlation function of this light is:

$$\langle E^*(t)E(t+\tau)\rangle = \nu\langle E_i^*(t)E_i(t+\tau)\rangle \quad ,$$

and, if there is no Doppler-broadening, the degree of first order coherence is:

$$g^{(1)}(\tau) = \frac{\langle E^*(t)E(t+\tau)\rangle}{\langle E^*(t)E(t)\rangle} = \exp\left(-\mathrm{i}\omega_0\tau - \gamma|\tau|\right) \quad .$$

Here, γ is the total damping rate. What is the degree of second order coherence for this light source as a function of the time delay τ? A hint is that you can assume that the number of atoms is very large, which means that some terms can be neglected.

3:9 Assume a large ensemble of small sources, that all radiate light with the angular frequency ω. Take the classical limit (i.e. no quantum mechanical effects) and sketch qualitatively the dependence of the first and second order degrees of coherence as a function of the time delay, τ. Do this for the two cases where the spectral broadening is dominated by collisional and Doppler broadening, respectively. For each of these cases, what determines the relevant time-scale for the variation (with τ) of $g^{(1)}(\tau)$ and $g^{(2)}(\tau)$?

Chapter 4

Quantization of the light field and of the interaction

4:1 Prove that

$$(n+3)(n+2)\langle n|\left(\hat{a}^\dagger\right)^3\hat{a}^4\hat{a}^\dagger|n\rangle = (n-1)(n-2)\langle n|\hat{a}^3\left(\hat{a}^\dagger\right)^4\hat{a}|n\rangle$$

4:2 Prove the commutators

$$\left[\hat{a},\left(\hat{a}^\dagger\right)^2\right] = 2\hat{a}^\dagger \qquad \text{and} \qquad \left[\hat{a}^2,\hat{a}^\dagger\right] = 2\hat{a}$$

and in general show that when n is a positive integer

$$\left[\hat{a},\left(\hat{a}^\dagger\right)^n\right] = 2\left(\hat{a}^\dagger\right)^{n-1} \qquad \text{and} \qquad \left[\hat{a}^n,\hat{a}^\dagger\right] = 2\hat{a}^{n-1}$$

Hence show that

$$\left[\hat{a},\exp\left(\beta\hat{a}^\dagger\right)\right] = \beta\exp\left[\beta\left(\hat{a}^\dagger\right)\right],$$

where the operator $\exp\left(\beta\hat{a}^\dagger\right)$ is defined in term of its power-series expansion.

4:3 Prove that the n th excited state of the oscillator can be expressed in terms of the ground state as

$$|n\rangle = \hat{N}(n)|0\rangle$$

where

$$\hat{N}(n) = \frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}}$$

4:4 Determine the mean number of thermally-excited photons from

$$\langle n\rangle = \mathrm{Tr}\left\{\hat{\rho}\hat{a}^\dagger\hat{a}\right\}$$

and hence show that the density operator for this case,

$$\hat{\rho} = \{1-\exp(-\hbar\omega/k_\mathrm{B}T)\}\sum_n \exp(-n\hbar\omega/k_\mathrm{B}T)|n\rangle\langle n|,$$

can be expressed as

$$\hat{\rho} = \sum_n \frac{\langle n\rangle^n}{(1+\langle n\rangle)^{1+n}}|n\rangle\langle n|.$$

4:5 Show that the density operator for single-mode thermal light,

$$\hat{\rho} = \{1 - \exp(-\hbar\omega/k_{\mathrm{B}}T)\} \sum_n \exp(-n\hbar\omega/k_{\mathrm{B}}T)|n\rangle\langle n|,$$

can be written in the equivalent form

$$\hat{\rho} = \{1 - \exp(-\hbar\omega/k_{\mathrm{B}}T)\} \sum_n \exp(-\hbar\omega\hat{a}^\dagger\hat{a}/k_{\mathrm{B}}T),$$

where the exponential is defined by its power-series expansion.

4:6 For a density operator

$$\hat{\rho} = \sum_{\{n_{\mathbf{k}p}\}} P\left(\{n_{\mathbf{k}p}\}\right) \left|\{n_{\mathbf{k}p}\}\right\rangle \left\langle\{n_{\mathbf{k}p}\}\right|$$

where

$$P\left(\{n_{\mathbf{k}p}\}\right) = \prod_{\mathrm{k}} \prod_p \frac{\langle n_{\mathbf{k}p}\rangle^n_{\mathbf{k}p}}{(1 + \langle n_{\mathbf{k}p}\rangle)^n_{\mathbf{k}p}}$$

prove

(a) The normalisation property, $\mathrm{Tr}\{\hat{\rho}\} = 1$.

(b) Show that the total mean number of photons is

$$\langle n\rangle = \sum_{\mathbf{k}} \sum_p \mathrm{Tr}\left\{\hat{\rho}\hat{a}^\dagger_{\mathbf{k}p}\hat{a}_{\mathbf{k}p}\right\} = \sum_{\mathbf{k}} \sum_p \langle n_{\mathbf{k}p}\rangle\,.$$

4:7 A polarized parallel light beam has electric field operators

$$\begin{aligned}
\hat{\mathbf{E}}^+_{\mathrm{T}}(\mathbf{r}, t) &= \sum_{\mathbf{k}} \sum_p \mathbf{e}_{\mathbf{k}p} \left(\hbar\omega_k/2\varepsilon_0 V\right)^{1/2} \hat{a}_{\mathbf{k}p} \exp\left[-\mathrm{i}\xi_{\mathbf{k}}(\mathbf{r}, t)\right] \\
\hat{\mathbf{E}}^-_{\mathrm{T}}(\mathbf{r}, t) &= \sum_{\mathbf{k}} \sum_p \mathbf{e}_{\mathbf{k}p} \left(\hbar\omega_k/2\varepsilon_0 V\right)^{1/2} \hat{a}^\dagger_{\mathbf{k}p} \exp\left[\mathrm{i}\xi_{\mathbf{k}}(\mathbf{r}, t)\right].
\end{aligned}$$

Show that the intensity operator

$$\hat{\mathbf{I}}(\mathbf{R}, t) = \varepsilon_0 c^2 \left\{\hat{\mathbf{E}}^-_{\mathrm{T}}(\mathbf{R}, t) \times \hat{\mathbf{B}}^+(\mathbf{R}, t) - \hat{\mathbf{B}}^-(\mathbf{R}, t) \times \hat{\mathbf{E}}^+_{\mathrm{T}}(\mathbf{R}, t)\right\}$$

is equivalent to an operator of magnitude

$$\hat{I}(\mathbf{R}, t) = 2\varepsilon_0 c \hat{E}^-_{\mathrm{T}}(\mathbf{R}, t)\hat{E}^+_{\mathrm{T}}(\mathbf{R}, t)$$

4:8 Use the second-quantized electric-dipole Hamiltonian in the interaction picture

$$\begin{aligned}
\mathcal{H}_{\mathrm{ED}}(t) = \mathrm{i}\sum_{\mathbf{k}} \sum_p \hbar g_{\mathbf{k}p} &\left\{\hat{\pi}^\dagger \hat{a}_{\mathbf{k}p} \exp\left[\mathrm{i}(\omega_0 - \omega_k)t + i\mathbf{k}\cdot\mathbf{R}\right]\right. \\
&\left. - \hat{a}^\dagger_{\mathbf{k}p}\hat{\pi} \exp\left[-\mathrm{i}(\omega_0 - \omega_k)t - i\mathbf{k}\cdot\mathbf{R}\right]\right\}
\end{aligned}$$

to work out the matrix-elements

$$\langle n_{\mathbf{k}p}, i|\mathcal{H}_{\mathrm{ED}}(t)|n'_{\mathbf{k}'p'}, j\rangle$$

where i, j denotes the atomic states (0 or 1); $n_{\mathbf{k}p}$ and $n'_{\mathbf{k}'p'}$ denotes the states of the radiation field.

Formulate the non-zero terms in form of easily understood diagrams.

4:9 Quantization of the electromagnetic field is often done by comparing a quantum mechanical harmonic oscillator with expressions for the classical field in the Coulomb gauge. The total energy of the classical field is a sum of the energies of each individual mode:

$$\mathcal{E}_{\mathrm{R}} = \sum_{\mathbf{k}} \sum_{p} \varepsilon_0 V \omega_k^2 \left(A_{\mathbf{k}p} A_{\mathbf{k}p}^* + A_{\mathbf{k}p}^* A_{\mathbf{k}p} \right) \quad ,$$

where $A_{\mathbf{k}p}$ is the modulus of the classical vector potential for the mode with wave-vector $\mathbf{k}$ and polarization p. The transversal part of the electric field is related to the vector potential through

$$\mathbf{E}_{\mathrm{T}}(\mathbf{r}, t) = -\frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} \quad .$$

The Hamiltonian for one mode of a quantum mechanical harmonic oscillator is

$$\widehat{\mathcal{H}}_{\mathbf{k}p} = \frac{1}{2} \hbar \omega_k \left(\hat{a}_{\mathbf{k}p} \hat{a}_{\mathbf{k}p}^\dagger + \hat{a}_{\mathbf{k}p}^\dagger \hat{a}_{\mathbf{k}p} \right) \quad ,$$

By doing the above mentioned comparison, derive an expression for the quantum mechanical transversal electric field operator.

Hints:
The classical vector potential $\mathbf{A}(\mathbf{r}, t)$ is a sum of contributions from all modes of the cavity:

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{k}} \sum_{p=1,2} \mathbf{e}_{\mathbf{k}p} A_{\mathbf{k}p}(\mathbf{r}, t) \quad ,$$

where $\mathbf{e}_{\mathbf{k}p}$ is a unit vector along the direction of polarization, and

$$A_{\mathbf{k}p}(\mathbf{r}, t) = A_{\mathbf{k}p} \exp\left(-\mathrm{i}\omega_k t + \mathrm{i}\mathbf{k} \cdot \mathbf{r}\right) + A_{\mathbf{k}p}^* \exp\left(\mathrm{i}\omega_k t - \mathrm{i}\mathbf{k} \cdot \mathbf{r}\right)$$

4:10 The second quantized electric-dipole interaction Hamiltonian can, in the Schrödinger picture, be written as

$$\widehat{\mathcal{H}}_{\mathrm{ED}} = \mathrm{i}e \sum_{\mathbf{k}} \sum_{p} \sum_{i,j} \sqrt{\hbar \omega_k / 2\varepsilon_0 V} \quad \frac{\mathbf{p}}{p} \cdot \mathbf{D}_{ij}$$
$$\left\{ \hat{a}_{\mathbf{k}p} \exp\left(\mathrm{i}\mathbf{k} \cdot \mathbf{R}\right) - \hat{a}_{\mathbf{k}p}^\dagger \exp\left(-\mathrm{i}\mathbf{k} \cdot \mathbf{R}\right) \right\} |i\rangle\langle j| \quad .$$

(a) Rewrite this for a two-level system, using the two-level transition operators $\hat{\pi}^\dagger$ and $\hat{\pi}$.

(b) What different terms will the final expression contain, and what physical processes do they represent?

4:11 In the interaction representation, the electric dipole interaction Hamiltonian for a two-level transition can be written as:

$$\widehat{\mathcal{H}}_{\mathrm{ED}}(t) = \mathrm{i}\sum_{\mathbf{k}}\sum_{p} \hbar g_{\mathbf{k}p}\{\hat{\pi}^{\dagger}\hat{a}_{\mathbf{k}p} \exp\left[\mathrm{i}(\omega_0 - \omega_k)t + \mathrm{i}\mathbf{k}\cdot\mathbf{R}\right]$$
$$-\hat{a}^{\dagger}_{\mathbf{k}p}\hat{\pi} \exp\left[-\mathrm{i}(\omega_0 - \omega_k)t - \mathrm{i}\mathbf{k}\cdot\mathbf{R}\right]\} \quad ,$$

where the constant $g_{\mathbf{k}p}$ includes constants and the projection of the dipole moment:

$$g_{\mathbf{k}p} = e\sqrt{\frac{\omega_k}{2\varepsilon_0\hbar V}} \quad \frac{\mathbf{p}_k}{p_k}\cdot\mathbf{D}_{12} \quad ,$$

and $\mathbf{p}_k$ is the polarization of the field. Assume single mode field. Absorption will now correspond to a transition from the state $|n,1\rangle$ to $|n-1,2\rangle$. Emission is instead a transition from $|n,2\rangle$ to $|n+1,1\rangle$. Show how the rates for absorption and emission depend on the number of photons in the mode. Interpret the meaning of different terms.

Chapter 5

Single-mode quantized light

5:1 Evaluate the expectation value $\langle n|(\hat{E}(\xi))^4|n\rangle$ and show that it exceeds the corresponding classical value with an amount 3/32, if the classical field amplitude is taken to be $E_0 = \sqrt{n+1/2}$.

5:2 An alternative approach to the coherent states is to take the eigenvalue equation

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

as the definition for a coherent state.

(a) Derive the expansion in terms of number states

$$|\alpha\rangle = \exp{(-\frac{1}{2}|\alpha|^2)} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

from this starting point.

(b) Show that the creation operator, $\hat{a}^\dagger$, has no right eigenstates.

5:3 A coherent state is defined by:

$$|\alpha\rangle = \exp{(-\frac{1}{2}|\alpha|^2)} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle \quad .$$

(a) What is the expectation value of the number of photons for this state and what is the quantum mechanical uncertainty?

(b) The electric field operator can be written in a dimensionless form as:

$$\hat{E}(\chi) = \hat{E}^+(\chi) + \hat{E}^-(\chi) = \frac{1}{2}\hat{a}\mathrm{e}^{-\mathrm{i}\chi} + \frac{1}{2}\hat{a}^\dagger\mathrm{e}^{\mathrm{i}\chi} = \hat{X}\cos\chi + \hat{Y}\sin\chi \quad ,$$

where $\hat{X}$ and $\hat{Y}$ are the "quadrature operators" and χ is the phase angle:

$$\chi = \omega t - kz - \frac{\pi}{2} \quad .$$

What is the "signal", the "noise" and the "signal-to-noise ratio", for a coherent state?

5:4 A coherent state is defined by:

$$|\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \quad .$$

(a) Show that the probabilty of finding n photons in the mode is

$$P(n) = e^{-\langle n\rangle}\frac{\langle n\rangle^n}{n!}$$

(b) Use Sterling's formula for the factorial to show that this is

$$P(n) \approx \frac{1}{\sqrt{2\pi\langle n\rangle}}\exp\left\{-\frac{(n-\langle n\rangle)^2}{2\langle n\rangle}\right\}$$

in what limit is this a good approximation?

5:5 Prove the operator relation

$$e^{-\hat{O}}\hat{a}e^{\hat{O}} = \hat{a} + \left[\hat{a},\hat{O}\right] + \frac{1}{2!}\left[\left[\hat{a},\hat{O}\right],\hat{O}\right] + \ldots$$

by power-series expansion of the exponentials. Hence show that

$$\hat{S}^\dagger(\zeta)\hat{a}\hat{S}(\zeta) = \hat{a}\cosh s - \hat{a}^\dagger e^{\mathrm{i}\vartheta}\sinh s$$

where the *squeeze* operator, $\hat{S}$ is

$$\hat{S}(\zeta) = \exp\left(\frac{1}{2}\zeta^*\hat{a}^2 - \frac{1}{2}\zeta\left(\hat{a}^\dagger\right)^2\right)$$

and ζ is the complex squeeze parameter

$$\zeta = s\exp(\mathrm{i}\vartheta)$$

5:6 The squeezed vacuum state is defined by

$$|\zeta\rangle = \hat{S}(\zeta)|0\rangle$$

(see problem **5:5** for definitions). Show

(a) that the mean-square number of photons is

$$\langle n^2\rangle = 3\sinh^4 s + 2\sinh^2 s = 3\langle n\rangle^2 + 2\langle n\rangle,$$

(b) the eigenvalue relation

$$(\hat{a}\cosh s - \hat{a}^\dagger e^{\mathrm{i}\vartheta}\sinh s)|\zeta\rangle = 0$$

5:7 The electric field operator can be written in dimensionless form as:

$$\hat{E}(\chi) = \hat{E}^{+}(\chi) + \hat{E}^{-}(\chi) = \frac{1}{2}\hat{a}e^{-i\chi} + \frac{1}{2}\hat{a}^{\dagger}e^{i\chi} = \hat{X}\cos\chi + \hat{Y}\sin\chi \quad ,$$

where $\hat{X}$ and $\hat{Y}$ are the "quadrature operators" and χ is the phase angle:

$$\chi = \omega t - kz - \frac{\pi}{2} \quad .$$

A single-mode quadrature squeezed state is defined by:

$$|\alpha, \zeta\rangle = \hat{D}(\alpha)\hat{S}(\zeta)|0\rangle \quad ,$$

where $\hat{D}(\alpha)$ is the coherent-state displacement operator, and $\hat{S}(\zeta)$ is the squeeze operator. ζ is the complex squeeze parameter with amplitude and phase defined by:

$$\zeta = s\exp(i\vartheta)$$

Some relations of these two operators are expressed in the following relations:

$$\hat{S}^{\dagger}(\zeta)\hat{D}^{\dagger}(\alpha)\hat{a}\hat{D}(\alpha)\hat{S}(\zeta) = \hat{a}\cosh s - \hat{a}^{\dagger}\exp(i\vartheta)\sinh s + \alpha$$

and

$$\hat{S}^{\dagger}(\zeta)\hat{D}^{\dagger}(\alpha)\hat{a}^{\dagger}\hat{D}(\alpha)\hat{S}(\zeta) = \hat{a}^{\dagger}\cosh s - \hat{a}\exp(-i\vartheta)\sinh s + \alpha^{*} \quad .$$

Derive the signal, the noise, and the signal-to-noise ratio for these squeezed coherent states. How do the results compare with an ordinary coherent state?

5:8 A single photon enters one of the input arms in a Mach-Zender interferometer. The other input is blocked. The two beam splitters in the interferometer are identical. The single-photon input state can be written as

$$|1\rangle_1|0\rangle_2 = \hat{a}_1^{\dagger}|0\rangle \quad .$$

(a) Write an expression for the state of one of the output arms.

(b) What is the expectation value of the number of photons in this output arm, expressed in the beam splitters reflection and transmission coefficients?

5:9 For arbitrary one-arm input (i.e., the input state is $|\text{arb}\rangle_1|0\rangle_2$) prove the following relations for the output states

$$\begin{aligned} \langle n_3(n_3-1)\rangle &= |\mathcal{R}|^4\langle n_1(n_1-1)\rangle \\ \langle n_4(n_4-1)\rangle &= |\mathcal{T}|^4\langle n_1(n_1-1)\rangle \\ \langle n_3 n_4\rangle &= |\mathcal{R}|^2|\mathcal{T}|^2\langle n_1(n_1-1)\rangle, \end{aligned}$$

hence show that

$$\left\langle (n_3-n_4)^2 \right\rangle = \left(|\mathcal{R}|^2 - |\mathcal{T}|^2\right)^2 \langle n_1(n_1-1)\rangle + \langle n_1\rangle$$

Chapter 6

Multi-mode quantized light

6:1 Derive the following expression for the first-order coherence of a parallel beam of thermal light

$$g^{(1)}(\tau) = \frac{\sum_k \omega_k \langle n_k \rangle \exp(-\mathrm{i}\omega_k \tau)}{\sum_k \omega_k \langle n_k \rangle}$$

This expression holds for all varieties of multimode light whose components have uncorrelated field amplitudes.

6:2 Prove the relation

$$g^{(2)}(\tau) = 1 + |g^{(1)}(\tau)|^2$$

for chaotic light, where the light is assumed to excite a large number of modes.

6:3 The continuos-mode creation and destruction operators fulfill

$$\left[\hat{a}(\omega), \hat{a}^\dagger(\omega')\right] = \delta(\omega - \omega')$$

(a) Derive the expressions for the electromagnetic field operators, $\hat{E}_{\mathrm{T}}^{+}$, $\hat{E}_{\mathrm{T}}^{-}$, $\hat{B}^{+}$ and $\hat{B}^{-}$ in terms of $\hat{a}(\omega)$ and $\hat{a}^\dagger(\omega)$. Explain any necessary assumptions.

(b) Show that the electromagnetic field Hamiltionan, defined from

$$\hat{\mathcal{H}}_{\mathrm{R}} = \int \mathrm{d}V \left[\varepsilon_0 \hat{\mathbf{E}}_{\mathrm{T}} \cdot \hat{\mathbf{E}}_{\mathrm{T}} + \mu_0^{-1} \hat{\mathbf{B}} \cdot \hat{\mathbf{B}}\right]$$

becomes

$$\hat{\mathcal{H}}_{\mathrm{R}} = \int_0^\infty \mathrm{d}\omega \hbar\omega \hat{a}^\dagger(\omega)\hat{a}(\omega) + \text{vacuum energy}$$

Again, explain any necessary assumptions.

6:4 Prove the commutation relations

$$\begin{aligned}\left[\hat{a}(\omega), \left(\hat{a}_\xi^\dagger\right)^n\right] &= n\xi(\omega)\left(\hat{a}_\xi^\dagger\right)^{n-1}, \\ \left[\hat{a}(t), \left(\hat{a}_\xi^\dagger\right)^n\right] &= n\xi(t)\left(\hat{a}_\xi^\dagger\right)^{n-1}.\end{aligned}$$

where

$$\hat{a}_\xi^\dagger = \int \mathrm{d}\omega \xi(\omega)\hat{a}^\dagger(\omega) = \int \mathrm{d}t \xi(t)\hat{a}^\dagger(t)$$

Also show that the continuous mode number state

$$|n_\xi\rangle = \frac{1}{\sqrt{n!}}\left(\hat{a}_\xi^\dagger\right)^n |0\rangle$$

is an eigenstate to the number-operator, i.e., show that

$$\hat{n}\,|n_\xi\rangle = n_\xi\,|n_\xi\rangle$$

where

$$\hat{n} = \int \mathrm{d}\omega \hat{a}^\dagger(\omega)\hat{a}(\omega) = \int \mathrm{d}t \hat{a}^\dagger(t)\hat{a}(t).$$

6:5 Explain in words, and if you want with a drawn figure, the meaning of "bunching" and "anti-bunching", and explain when these phenomena occur. What is the connection between these concepts and the second order coherence? Where does chaotic and coherent light fit into this? Does these phenomena have classical analogues, or are any of them purely quantum mechanical?

6:6 A photon pair state is created that excites two different continuous-mode fields. These two photons are arranged such that they excite the two different input arms in a beam splitter. Thus, the input state of the beam splitter can be written as

$$|(1_1, 1_2)_\beta\rangle = \int\int \beta(t,t')\hat{a}_1^\dagger(t)\hat{a}_2^\dagger(t')\mathrm{d}t\mathrm{d}t' \quad . \tag{6.1}$$

The reflectance of the beam splitter is $|\mathcal{R}|^2$ and the transmittance is $|\mathcal{T}|^2$. $\mathcal{R}$ and $\mathcal{T}$ are respectively the complex reflection and transmission coefficients.

(a) Rewrite eq. (6.1) in a form where it contains the creation operators for the output arms instead of those for the input arms.

(b) It is a 50:50 beam splitter, which means that the reflectance and the transmittance are equal ($|\mathcal{R}|^2 = |\mathcal{T}|^2 = 1/2$). Moreover, the two states overlap perfectly in time. In other words, the joint overlap integral is unity:

$$|J|^2 = \int\int \beta^*(t,t')\beta(t',t)\mathrm{d}t\mathrm{d}t' = 1 \quad .$$

For this particular case, the probability is for finding one photon in each of the two output arms at a measurement is zero. Interpret this result. What does it mean, and why does this happen?

6:7 The photon pair-state creation operator is

$$\hat{P}^{\dagger}_{\beta aa} = \frac{1}{\sqrt{2}} \int \mathrm{d}\omega \int \mathrm{d}\omega' \beta(\omega, \omega') \hat{a}^{\dagger}(\omega) \hat{a}^{\dagger}(\omega')$$

show that the mean photon flux is

$$f(t) = 2 \int \mathrm{d}t' |\beta(t, t')|^2 .$$

6:8 Assume that a pair state is fed into a beam splitter, in a ways such that one photon excites one of the input arms, and the other photon excites the other. Show what the probability is that one will detect both photons in the same output arm, and what the probability is that one instead will detect one photon in each output arm. Explain all the steps in your derivation.

How can one interpret the result?

6:9 The *Casimir force* can be seen as a reduction of the vacuum energy. Derive the expression for the Casimir force acting on two large parallell plates, with area A, of perfect conductors, at a distance a from each other.

(a) The possible wave-vectors in the z direction is

$$k_n = \frac{n\pi}{a}$$

leading to possible frequencies

$$\omega_n = c\sqrt{k_x^2 + k_y^2 + \frac{n^2\pi^2}{a^2}}, \quad \text{for } n = 1, 2, \ldots$$

show that the vacuum-energy becomes

$$\frac{\mathcal{E}_{\text{cav}}(a)}{A} = \frac{\hbar}{2} 2 \int \frac{\mathrm{d}k_x \mathrm{d}k_y}{(2\pi)^2} \sum_{n=1}^{\infty} \omega_n$$

(b) Show that the vacuum-energy of the same region in the absence of one the plates can be written as

$$\frac{\mathcal{E}_{\text{free}}(a)}{A} = \frac{\hbar}{2} 2 \int \frac{\mathrm{d}k_x \mathrm{d}k_y}{(2\pi)^2} \int_0^{\infty} \mathrm{d}n \omega_n$$

(with ω_n given by the same expression as before.)
Hint: Imagine that the plates are L apart and let L go to infinity.

(c) Rewrite the integral using polar coordinates to show that

$$\frac{\mathcal{E}_{\text{cav}}(a)}{A} = \frac{\hbar c}{2\pi} \sum_{n=1}^{\infty} \int_0^{\infty} q \mathrm{d}q w_n(q)$$

and

$$\frac{\mathcal{E}_{\text{free}}(a)}{A} = \frac{\hbar c}{2\pi} \int_0^{\infty} \mathrm{d}n \int_0^{\infty} q \mathrm{d}q w_n(q)$$

where

$$w_n(q) = \sqrt{q^2 + \frac{n^2\pi^2}{a^2}}$$

(d) Introduce a regularization parameter μ so that these expressions becomes

$$\frac{\mathcal{E}_{\text{cav}}(a)}{A} = \frac{\hbar c}{2\pi} \sum_{n=1}^{\infty} \int_0^{\infty} q \mathrm{d}q w_n(q) e^{-\mu w_n(q)}$$

and

$$\frac{\mathcal{E}_{\text{free}}(a)}{A} = \frac{\hbar c}{2\pi} \int_0^{\infty} \mathrm{d}n \int_0^{\infty} q \mathrm{d}q w_n(q) e^{-\mu w_n(q)}$$

show that

$$\frac{\delta\mathcal{E}}{A} \equiv \lim_{\lambda \to 0} \left(\frac{\mathcal{E}_{\text{cav}}(a)}{A} - \frac{\mathcal{E}_{\text{free}}(a)}{A} \right) = -\frac{\hbar c \pi^2}{3 \cdot 240 a^3}$$

and thus that the pressure generated is

$$\mathcal{P} = -\frac{\hbar c \pi^2}{240 a^4}$$

(e) Consider two 10 nm thick plates of gold, at what distance is the Casimir force equal to gravitanional force? equal to 1 atm? Could this have any consequences for nanoelectromechanical systems (NEMS)?

Chapter 7

Absorption and amplification of light by matter

7:1 Derive the mean -photon-number rate-equation

$$\frac{\mathrm{d}\langle n\rangle}{\mathrm{d}t} = \Gamma_{\mathrm{st}}\left\{N_2 + (N_2 - N_1)\langle n\rangle\right\}$$

7:2 Derive an equation of motion for the second factorial moment of the photon-number distribution in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\langle n(n-1)\rangle - 2\langle n\rangle^2\right] = -2\Gamma_{\mathrm{st}}(N_1 - N_2)\left[\langle n(n-1)\rangle - 2\langle n\rangle^2\right]$$

where N_1 and N_2 are the solutions to

$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = -\frac{\mathrm{d}N_2}{\mathrm{d}t} = N_2\Gamma_{\mathrm{sp}} + (N_2 - N_1)\Gamma_{\mathrm{st}}\langle n\rangle$$

7:3 Prove that

$$P(n) = \sum_{l=n}^{\infty} P_0(l)\frac{l!}{(l-n)!n!}(1-p)^{l-n}p^n$$

with

$$p = \exp(-N\Gamma_{\mathrm{st}}t)$$

is a solution to

$$\frac{\mathrm{d}P(n)}{\mathrm{d}t} = N\Gamma_{\mathrm{st}}t\left[-nP(n) + (n+1)P(n+1)\right].$$

7:4 A three-level system, as in fig. 7.1 is used in order realize a laser.

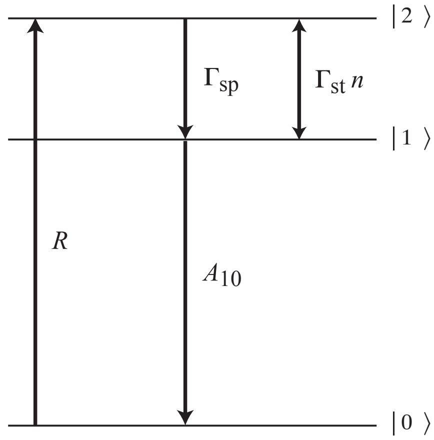

Figure 7.1: Atomic energy-level scheme for a three-level laser showing the relevant transition rates.

The rates indicated in the figure are the rate of pumping from $|0\rangle$ to $|2\rangle$, R ($|0\rangle \leftrightarrow |2\rangle$ is dipole forbidden, so this pumping is done with for example electron bombardment); the spontaneous decay from $|1\rangle$ to $|0\rangle$, A_{10}; the spontaneous emission rate from $|2\rangle$ to $|1\rangle$, Γ_{sp}; and the rates for absorption and stimulated emission between states $|1\rangle$ and $|2\rangle$, $\Gamma_{\mathrm{st}}n$. Another important rate is the loss of cavity photons due to reflections in the output mirrors, Γ_{cav}. Of all these rates, A_{10} is by far greater than all the other.

(a) Draw an energy-level diagram for the photons in the lasing mode. Indicate by arrows and symbols the relevant transitions that contribute to the probability of having n photons in the lasing mode.

(b) Write down rate equations for the probability of having n photons in the lasing mode, and for the population in state $|2\rangle$.

(c) What is the population of state $|2\rangle$ at steady-state? How does it depend on n, and how can that be interpreted physically?

7:5 Prove that the above-threshold normalized photon-number distribution is given by

$$P(n) = \frac{\exp\left[-(n - \langle n\rangle)^2/2(n_s + \langle n\rangle)\right]}{\left[2\pi(n_s + \langle n\rangle)\right]^2}$$

to a very good approximation.

7:6 Prove that the output signal operators defined by

$$\hat{a}_L(\omega) = \exp\left\{\left[\mathrm{i}\eta(\omega)(\omega/c) - \frac{1}{2}K(\omega)\right] L\right\} \hat{a}_0(\omega)$$
$$+\mathrm{i}\sqrt{K(\omega)} \int_0^L \mathrm{d}z \exp\left\{\left[\mathrm{i}\eta(\omega)(\omega/c) - \frac{1}{2}K(\omega)\right](L - z)\right\} \hat{b}(z, \omega)$$

fullfils the standard commutation relation

$$\left[\hat{a}_L(\omega), \hat{a}_L^\dagger(\omega')\right] = \delta(\omega - \omega')$$

7:7 The output noise is represented by the operator $\hat{b}_{\mathcal{N}}$, defined by

$$\hat{b}_{\mathcal{N}}(\omega) = \mathrm{i}\sqrt{G(\omega)} \int_0^L \mathrm{d}z \exp\left\{\left[\mathrm{i}\eta(\omega)(\omega/c) + \frac{1}{2}G(\omega)\right](L-z)\right\} \hat{b}^\dagger(z,\omega)$$

so that

$$\hat{a}_L(\omega) = \exp\left\{\left[\mathrm{i}\eta(\omega)(\omega/c) + \frac{1}{2}G(\omega)\right]L\right\} \hat{a}_0(\omega) + \hat{b}_{\mathcal{N}}(\omega)$$

show that

$$\left[\hat{b}_{\mathcal{N}}(\omega), \hat{b}^\dagger_{\mathcal{N}}(\omega')\right] = -\left[\exp(G(\omega)L) - 1\right]\delta(\omega - \omega')$$

7:8 The equation of motion for $\hat{a}_{\mathbf{k}p}$ can be formally integrated to give

$$\hat{a}_{\mathbf{k}p}(t) = \exp(-\mathrm{i}\omega_k t)\left\{\hat{a}_{\mathbf{k}p}(0) - g_{\mathbf{k}p}\int_0^t \mathrm{d}t'\hat{\pi}(t')\exp(-\mathrm{i}\mathbf{k}\cdot\mathbf{R} + \mathrm{i}\omega_k t')\right\}$$

show that the commutation relation

$$\left[\hat{a}_{\mathbf{k}p}(t), \hat{a}^\dagger_{\mathbf{k}'p'}(t)\right] = \delta_{\mathbf{k},\mathbf{k}'}\delta_{p,p'}$$

is fullfilled to order $g^2_{\mathbf{k}p}$.

7:9 The source-field expression expresses the transversal electric field radiated by an excited atom. The modulus of this field is

$$\hat{E}_{\mathrm{sf}} = \hat{E}^+_{\mathrm{sf}} + \hat{E}^-_{\mathrm{sf}} = -\frac{e\omega_0^2 D_{12}\sin\Theta}{4\pi\varepsilon_0 c^2|\mathbf{r}-\mathbf{R}|} \quad \hat{\pi}(t - \frac{|\mathbf{r}-\mathbf{R}|}{c}) + \mathrm{h.c.}$$

where D_{12} and Θ are the magnitude and the direction of the atomic dipole moment, $\mathbf{R}$ is the position of the atom, and $\hat{\pi}$ is the atomic transition operator, $|1\rangle\langle 2|$.The general expressions for the first and second order coherence are

$$g^{(1)}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = \frac{\left\langle \hat{E}^-_T(\mathbf{r}_1,t_1)\hat{E}^+_T(\mathbf{r}_2,t_2)\right\rangle}{\sqrt{\left\langle \hat{E}^-_T(\mathbf{r}_1,t_1)\hat{E}^+_T(\mathbf{r}_1,t_1)\right\rangle \left\langle \hat{E}^-_T(\mathbf{r}_2,t_2)\hat{E}^+_T(\mathbf{r}_2,t_2)\right\rangle}}$$

$$g^{(2)}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = \frac{\left\langle \hat{E}^-_T(\mathbf{r}_1,t_1)\hat{E}^-_T(\mathbf{r}_2,t_2)\hat{E}^+_T(\mathbf{r}_2,t_2)\hat{E}^+_T(\mathbf{r}_1,t_1)\right\rangle}{\left\langle \hat{E}^-_T(\mathbf{r}_1,t_1)\hat{E}^+_T(\mathbf{r}_1,t_1)\right\rangle \left\langle \hat{E}^-_T(\mathbf{r}_2,t_2)\hat{E}^+_T(\mathbf{r}_2,t_2)\right\rangle}.$$

(a) Use this in order to obtain the two orders of coherence for the emission from an excited atom. The expression should only contain atomic transition operators.

(b) What are the values of these two orders of coherence for zero time-delay? Interpret that result very briefly.

7:10 Use the quantum theory of direct detection to show that the measured degree of second-order coherence of the light radiated by a single driven atom for zero time delay is

$$g_{\mathrm{D}}^{(2)}(0) = \frac{1 - 2\gamma_{\mathrm{sp}}T + 2\gamma_{\mathrm{sp}}^2 T^2 - \exp(-2\gamma_{\mathrm{sp}}T)}{2\gamma_{\mathrm{sp}}^2 T^2}.$$

Chapter 8

Scattering of light by matter and light induced fluorescence

8:1 Here is the first problem.

8:2 Here is an itemized problem:

(a) this the first part of the question.
(b) this is the second,
(c) and this is the third.

8:3 This is the third problem.

8:4 Here is a problem that includes a figure, except that that part has been commented away.

8:5 Here is a problem that contains an equation:

$$F(t) \propto \frac{1}{\tau_0} N_{S_1}(t)$$