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EXAM

Växelverkan mellan ljus och materia — Quantum Optics, 5p
2006–05–31, at 9.00–15.00
Östra paviljongen, room 5 **(bokningskod: 86154)**

Allowed aids:

- Physics Handbook (Nordling/Österman)
- Beta - Mathematical handbook
- Pocket calculator
- Three A4 pages of handwritten notes (with text on both sides, but not including solved problems and examples).

Every problem will give a maximun of 1.00 points. The calculations and the reasoning must always be fully accounted for in a way that is easy to follow. Write your name on *all* submitted papers.

Good luck!

1 In the semi-classical approximation, the optical Bloch equations for a two-level system can be written as:

$$\frac{\mathrm{d}\tilde{\rho}_{22}}{\mathrm{d}t} = -\frac{\mathrm{d}\tilde{\rho}_{11}}{\mathrm{d}t} = -\frac{\mathrm{i}\Omega}{2}(\tilde{\rho}_{12} - \tilde{\rho}_{21}) - 2\gamma_{\mathrm{sp}}\tilde{\rho}_{22} \tag{1}$$

$$\frac{\mathrm{d}\tilde{\rho}_{12}}{\mathrm{d}t} = \frac{\mathrm{d}\tilde{\rho}^*_{21}}{\mathrm{d}t} = \frac{\mathrm{i}\Omega}{2}(\tilde{\rho}_{11} - \tilde{\rho}_{22}) + [\mathrm{i}(\omega_0 - \omega) - \gamma_{\mathrm{sp}}]\,\tilde{\rho}_{12}\,. \tag{2}$$

The general solution for the excited state population is of the form:

$$\tilde{\rho}_{22}(t) = \tilde{\rho}_{22}(\infty)\left\{1 + [a\cos(\lambda t) + b\sin(\lambda t)]\exp\left(-\frac{3\gamma_{\mathrm{sp}}t}{2}\right)\right\}, \tag{3}$$

where

$$\tilde{\rho}_{22}(\infty) = \frac{\Omega^2/2}{2\gamma_{\mathrm{sp}}^2 + \Omega^2} \quad , \quad \lambda = \sqrt{\Omega^2 - \gamma_{\mathrm{sp}}^2/4} \quad . \tag{4}$$

For the initial conditions that the atom is purely in the ground state at $t = 0$, and a resonant field is suddenly turned on, the coefficients a and b are:

$$a = -1 \quad , \quad b = -\frac{3\gamma_{\mathrm{sp}}}{2\lambda}\,. \tag{5}$$

Assume that the two-level atom has been prepared (infinitely fast) in a normalized equal superposition of the ground and the excited state at $t = 0$, as:

$$\Psi(\mathbf{r}, 0) = \frac{1}{\sqrt{2}}(\psi_1(\mathbf{r}) + \psi_2(\mathbf{r}))\,. \tag{6}$$

The Einstein A coefficient of the transition is $A = 3.2 \times 10^6\ \mathrm{s}^{-1}$, and the resonant transition wavelength is $\lambda = 780$ nm. The atom is exposed to a pulse of single-mode resonant light between $t = 0$ and $t = \tau$, with an irradiance of $I = 46\ \mathrm{mW/cm}^2$.

a. To what extent is it possible to adjust τ such that the atom after this interaction is almost entirely in the excited state, and to what extent can the corresponding thing be done for the ground state (*i.e.* how large can the respective populations become)?

b. How long should these times be?

Hints: Assume a Clebsch-Gordan coefficient of one, *i.e.* that the atomic dipole moment is parallel with the polarization of the field. When determining the proper form for the time evolution for a certain initial condition, both $\tilde{\rho}_{22}$ and its time derivative at $t = 0$ are needed. The following relations may be useful (with standard definitions for the involved symbols):

$$2\gamma_{\mathrm{sp}} = A_{21} = \tau_{\mathrm{R}}^{-1} = \frac{e^2\omega_0^3 D_{12}^2}{3\pi\varepsilon_0\hbar c^3} \quad , \quad \hbar\Omega = eE_0(\mathbf{e}\cdot\mathbf{D}_{12}) \tag{7}$$

2 A three-level system, as in figure 1 is used in order realize a laser.

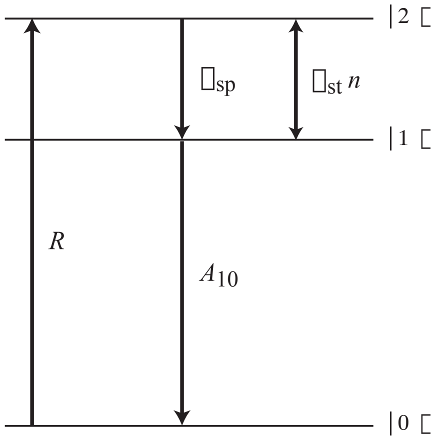

Figure 1: Atomic energy-level scheme for a three-level laser showing the relevant transition rates.

The rates indicated in the figure are; the rate of pumping from $|0\rangle$ to $|2\rangle$, R ($|0\rangle \leftrightarrow |2\rangle$ is dipole forbidden, so this pumping is done with for example electron bombardment); the spontaneous decay from $|1\rangle$ to $|0\rangle$, A_{10}; the spontaneous emission rate from $|2\rangle$ to $|1\rangle$, Γ_{sp}; and the rates for absorption and stimulated emission between states $|1\rangle$ and $|2\rangle$, $\Gamma_{\text{st}}n$. Another important rate is the loss of cavity photons due to imperfect reflections in the output mirrors, Γ_{cav}. Of all these rates, A_{10} is by far greater than all the others.

a. Draw an energy-level diagram for the the lasing mode, involving numbers of photons. Indicate by arrows and symbols the relevant transitions that contribute to the probability of having n photons in the lasing mode.

b. Write down rate equations for the probability of having n photons in the lasing mode, and for the population in state $|2\rangle$.

c. What is the population of state $|2\rangle$ at steady-state? How does it depend on n, and how can that be interpreted physically?

3 A single photon enters one of the input arms in a Mach-Zender interferometer. The other input is blocked. The two beam splitters in the interferometer are identical. The single-photon input state can be written as:

$$|1\rangle_1|0\rangle_2 = \hat{a}_1^\dagger|0\rangle\,. \tag{8}$$

What is the expectation value of the number of photons in either output arm, expressed in the beam splitters reflection and transmission coefficients?

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4 The quantum mechanical electric field operator for a single-mode, can be described as follows:

$$\widehat{\mathbf{E}}_{\mathrm{T}}(\mathbf{r},t) = \widehat{\mathbf{E}}_{\mathrm{T}}^{+}(\mathbf{r},t) + \widehat{\mathbf{E}}_{\mathrm{T}}^{-}(\mathbf{r},t)\,, \tag{9}$$

with the definitions:

$$\begin{aligned}
\widehat{\mathbf{E}}_{\mathrm{T}}^{+}(\mathbf{r},t) &= \sum_{\mathbf{k}}\sum_{p}\mathbf{e}_{\mathbf{k}p}\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}\,\hat{a}_{\mathbf{k}p}\,\exp\left[-\mathrm{i}\chi_{\mathbf{k}}(\mathbf{r},t)\right]\\
\widehat{\mathbf{E}}_{\mathrm{T}}^{-}(\mathbf{r},t) &= \sum_{\mathbf{k}}\sum_{p}\mathbf{e}_{\mathbf{k}p}\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}\,\hat{a}_{\mathbf{k}p}^{\dagger}\,\exp\left[\mathrm{i}\chi_{\mathbf{k}}(\mathbf{r},t)\right]\\
\chi_{\mathbf{k}}(\mathbf{r},t) &= \omega_k t - \mathbf{k}\cdot\mathbf{r} - \pi/2 \quad .
\end{aligned}$$

From this, derive an expression for the electric field operator for a multimode field, with a continuous distribution of frequencies. For simplicity, assume a single polarization so that the vector properties of the field can be ignored.

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5 Figure 2 shows the rates of absorption, stimulated emission and spontaneous emission as a function of the mean value of the energy density of the radiation. The curves are derived using only the phenomenological Einstein theory and rate equations for a two-level system. No quantum mechanics have been used. The energy density is given in units of the 'saturation energy density'.

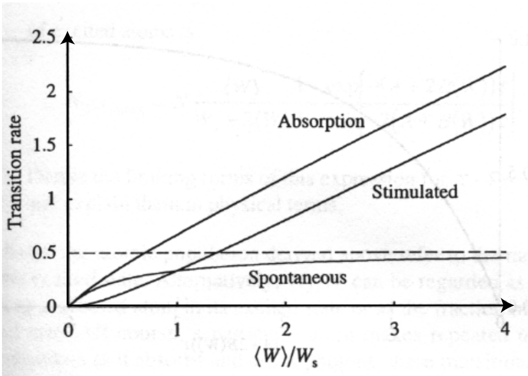

Figure 2: Mean rates of the three Einstein transitions in units of the A coefficient as functions of the radiative energy density.

Make an interpretation of the curves. What does it mean that they look like they do? How will the corresponding populations in the upper and the lower state depend on the energy density? Suppose the radiation field is called the pump field. How will an extra applied weak beam of resonant light be affected if it interacts with this system, under the conditions of high and low energy density respectively for the pump field?