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EXAM

Växelverkan mellan ljus och materia - Quantum Optics, 5p
2004–09–29, at 9.00–15.00, Dept. of Physics, seminar room .

Allowed: Physics handbook, Beta, pocket calculator and three A4 pages of handwritten notes (with text on both sides, but not including solved problems and examples).

The calculations and the reasoning should be easy to follow. *Good luck!*

1 Consider the classical electric field of a plane parallel light beam, made up of independent contributions from a large ensemble of radiating atoms:

$$E(t) = \sum_{i=1}^{\nu} E_i(t) \quad , \tag{1}$$

where ν is the number of atoms. The first order electric field correlation function of this light is:

$$\langle E^*(t)E(t+\tau)\rangle = \nu \langle E_i^*(t)E_i(t+\tau)\rangle \quad , \tag{2}$$

and, if there is no Doppler-broadening, the degree of first order coherence is:

$$g^{(1)}(\tau) = \frac{\langle E^*(t)E(t+\tau)\rangle}{\langle E^*(t)E(t)\rangle} = \exp\left(-\mathrm{i}\omega_0\tau - \gamma|\tau|\right) \quad . \tag{3}$$

Here, γ is the total damping rate. What is the degree of second order coherence for this light source as a function of the time delay τ? A hint is that you can assume that the number of atoms is very large, which means that some terms can be neglected.

2 Quantization of the electromagnetic field is often done by comparing a quantum mechanical harmonic oscillator with expressions for the classical field in the Coulomb gauge. The total energy of the classical field is a sum of the energies of each individual mode:

$$\mathcal{E}_{\mathrm{R}} = \sum_{\mathbf{k}} \sum_{p} \varepsilon_0 V \omega_k^2 \left(A_{\mathbf{k}p} A_{\mathbf{k}p}^* + A_{\mathbf{k}p}^* A_{\mathbf{k}p} \right) \quad , \tag{4}$$

where $A_{\mathbf{k}p}$ is the modulus of the classical vector potential for the mode with wave-vector $\mathbf{k}$ and polarization p. The transversal part of the electric field is related to the vector potential through

$$\mathbf{E}_{\mathrm{T}}(\mathbf{r}, t) = -\frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} \quad . \tag{5}$$

The Hamiltonian for one mode of a quantum mechanical harmonic oscillator is

$$\widehat{\mathcal{H}}_{\mathbf{k}p} = \frac{1}{2} \hbar \omega_k \left(\hat{a}_{\mathbf{k}p} \hat{a}_{\mathbf{k}p}^\dagger + \hat{a}_{\mathbf{k}p}^\dagger \hat{a}_{\mathbf{k}p} \right) \quad , \tag{6}$$

By doing the above mentioned comparison, derive an expression for the quantum mechanical transversal electric field operator.

Hints:
The classical vector potential $\mathbf{A}(\mathbf{r}, t)$ is a sum of contributions from all modes of the cavity:

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{k}} \sum_{p=1,2} \mathbf{e}_{\mathbf{k}p} A_{\mathbf{k}p}(\mathbf{r}, t) \quad , \tag{7}$$

where $\mathbf{e}_{\mathbf{k}p}$ is a unit vector along the direction of polarization, and

$$A_{\mathbf{k}p}(\mathbf{r}, t) = A_{\mathbf{k}p} \exp\left(-\mathrm{i}\omega_k t + \mathrm{i}\mathbf{k} \cdot \mathbf{r}\right) + A_{\mathbf{k}p}^* \exp\left(\mathrm{i}\omega_k t - \mathrm{i}\mathbf{k} \cdot \mathbf{r}\right) \tag{8}$$

3 A three-level system, as in fig. **??** is used in order realize a laser.

The rates indicated in the figure are the rate of pumping from $|0\rangle$ to $|2\rangle$, R ($|0\rangle \leftrightarrow |2\rangle$ is dipole forbidden, so this pumping is done with for example electron bombardment); the spontaneous decay from $|1\rangle$ to $|0\rangle$, A_{10}; the spontaneous emission rate from $|2\rangle$ to $|1\rangle$, Γ_{sp}; and the rates for absorption and stimulated emission between states $|1\rangle$ and $|2\rangle$, $\Gamma_{\mathrm{st}}n$. Another important rate is the loss of cavity photons due to imperfect reflections in the output mirrors, Γ_{cav}. Of all these rates, A_{10} is by far greater than all the others.

3a) Draw an energy-level diagram for the the lasing mode, involving numbers of photons. Indicate by arrows and symbols the relevant transitions that contribute to the probability of having n photons in the lasing mode.

3b) Write down rate equations for the probability of having n photons in the lasing mode, and for the population in state $|2\rangle$.

3c) What is the population of state $|2\rangle$ at steady-state? How does it depend on n, and how can that be interpreted physically?

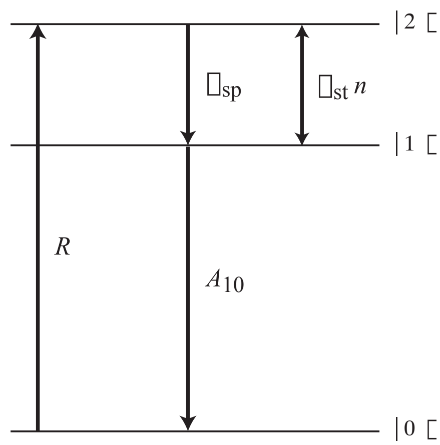

Figure 1: Atomic energy-level scheme for a three-level laser showing the relevant transition rates.

4 The electric field operator can be written in dimensionless form as:

$$\hat{E}(\chi) = \hat{E}^{+}(\chi) + \hat{E}^{-}(\chi) = \frac{1}{2}\hat{a}\mathrm{e}^{-\mathrm{i}\chi} + \frac{1}{2}\hat{a}^{\dagger}\mathrm{e}^{\mathrm{i}\chi} = \hat{X}\cos\chi + \hat{Y}\sin\chi \quad , \tag{9}$$

where $\hat{X}$ and $\hat{Y}$ are the ”quadrature operators” and χ is the phase angle:

$$\chi = \omega t - kz - \frac{\pi}{2} \quad . \tag{10}$$

A single-mode quadrature squeezed state is defined by:

$$|\alpha, \zeta\rangle = \hat{D}(\alpha)\hat{S}(\zeta)|0\rangle \quad , \tag{11}$$

where $\hat{D}(\alpha)$ is the coherent-state displacement operator, and $\hat{S}(\zeta)$ is the squeeze operator. ζ is the complex squeeze parameter with amplitude and phase defined by:

$$\zeta = s\exp(\mathrm{i}\vartheta) \tag{12}$$

Some relations of these two operators are expressed in the following relations:

$$\hat{S}^{\dagger}(\zeta)\hat{D}^{\dagger}(\alpha)\hat{a}\hat{D}(\alpha)\hat{S}(\zeta) = \hat{a}\cosh s - \hat{a}^{\dagger}\exp(\mathrm{i}\vartheta)\sinh s + \alpha \tag{13}$$

and

$$\hat{S}^{\dagger}(\zeta)\hat{D}^{\dagger}(\alpha)\hat{a}^{\dagger}\hat{D}(\alpha)\hat{S}(\zeta) = \hat{a}^{\dagger}\cosh s - \hat{a}\exp(-\mathrm{i}\vartheta)\sinh s + \alpha^{*} \quad . \tag{14}$$

Derive the signal, the noise, and the signal-to-noise ratio for these squeezed coherent states. How do the results compare with an ordinary coherent state?

5 What is the *Casimir force*? When, where and why does it occur and is it attractive or repulsive? You do not need to quantify the force, but I want a fairly detailed qualitative discussion explaining the origin of the force.