Umeå Universitet
Fysiska Institutionen
Anders Kastberg

EXAM

Växelverkan mellan ljus och materia - Quantum Optics, 5p
2004–06–02, at 9.00–15.00, Östra paviljongen, room 3.

Allowed: Physics handbook, Beta, pocket calculator and three A4 pages of handwritten notes (with text on both sides, but not including solved problems and examples).

The calculations and the reasoning should be easy to follow. *Good luck!*

1 The "Kramers-Heisenberg formula" for the differential cross-section is

$$\frac{\mathrm{d}\sigma(\omega)}{\mathrm{d}\Omega} = \sum_{f}^{\omega_f \leq \omega} \frac{e^4 \omega (\omega - \omega_f)^3}{16\pi^2 \varepsilon_0^2 \hbar^2 c^4} \cdot \left| \sum_l \left(\frac{(\mathbf{e}_{\mathrm{sc}} \cdot \mathbf{D}_{fl})(\mathbf{e} \cdot \mathbf{D}_{li})}{\omega_l - \omega} + \frac{(\mathbf{e} \cdot \mathbf{D}_{fl})(\mathbf{e}_{\mathrm{sc}} \cdot \mathbf{D}_{li})}{\omega_l - \omega_{\mathrm{sc}}} \right) \right|^2 \tag{1}$$

Here, the zero-point in energy is taken as the energy of the initial state $|i\rangle$. The energy of the final state $|f\rangle$ is $\hbar\omega_f$ and $\hbar\omega_l$ is the energy of intermediate states $|l\rangle$. ω and ω_{sc} are the angular frequencies of the incident and the scattered light respectively. The various vectors $\mathbf{e}$ and $\mathbf{D}$ are the relevant unit polarization vectors and the dipole moments.

1a) Explain what this equation tells us and when it can be used.

1b) Consider the case when the scatterers are the various constituents of air and the incident light includes the whole visible spectrum. For all important atmospherical particles, the lowest excitation energies correspond to frequency in the ultra-violet of higher. Moreover, we consider only the Rayleigh scattered part of the cross-section. How can eq. (1) be simplified for this case? Explain the consequence of the formula you have derived for the frequency spectrum of skylight.

2 Quantization of the electromagnetic field is often done by comparing a quantum mechanical harmonic oscillator with expressions for the classical field in the Coulomb gauge. The total energy of the classical field is a sum of the energies of each individual mode:

$$\mathcal{E}_{\mathrm{R}} = \sum_{\mathbf{k}} \sum_{p} \varepsilon_0 V \omega_k^2 \left(A_{\mathbf{k}p} A_{\mathbf{k}p}^* + A_{\mathbf{k}p}^* A_{\mathbf{k}p} \right) \quad , \tag{2}$$

where $A_{\mathbf{k}p}$ is the modulus of the classical vector potential for the mode with wave-vector $\mathbf{k}$ and polarization p. The transversal part of the electric field is related to the vector potential through

$$\mathbf{E}_{\mathrm{T}}(\mathbf{r}, t) = -\frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} \quad . \tag{3}$$

The Hamiltonian for one mode of a quantum mechanical harmonic oscillator is

$$\widehat{\mathcal{H}}_{\mathbf{k}p} = \frac{1}{2} \hbar \omega_k \left(\hat{a}_{\mathbf{k}p} \hat{a}_{\mathbf{k}p}^\dagger + \hat{a}_{\mathbf{k}p}^\dagger \hat{a}_{\mathbf{k}p} \right) \quad , \tag{4}$$

By doing the above mentioned comparison, derive an expression for the quantum mechanical transversal electric field operator.

Hints:
The classical vector potential $\mathbf{A}(\mathbf{r}, t)$ is a sum of contributions from all modes of the cavity:

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{k}} \sum_{p=1,2} \mathbf{e}_{\mathbf{k}p} A_{\mathbf{k}p}(\mathbf{r}, t) \quad , \tag{5}$$

where $\mathbf{e}_{\mathbf{k}p}$ is a unit vector along the direction of polarization, and

$$A_{\mathbf{k}p}(\mathbf{r}, t) = A_{\mathbf{k}p} \exp\left(-\mathrm{i}\omega_k t + \mathrm{i}\mathbf{k} \cdot \mathbf{r}\right) + A_{\mathbf{k}p}^* \exp\left(\mathrm{i}\omega_k t - \mathrm{i}\mathbf{k} \cdot \mathbf{r}\right) \tag{6}$$

3 A coherent state is defined by:

$$|\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \quad . \tag{7}$$

3a) What is the expectation value of the number of photons for this state and what is the quantum mechanical uncertainty?

3b) The electric field operator can be written in a dimensionless form as:

$$\hat{E}(\chi) = \hat{E}^{+}(\chi) + \hat{E}^{-}(\chi) = \frac{1}{2}\hat{a}\mathrm{e}^{-\mathrm{i}\chi} + \frac{1}{2}\hat{a}^{\dagger}\mathrm{e}^{\mathrm{i}\chi} = \hat{X}\cos\chi + \hat{Y}\sin\chi \quad , \tag{8}$$

where $\hat{X}$ and $\hat{Y}$ are the ”quadrature operators” and χ is the phase angle:

$$\chi = \omega t - kz - \frac{\pi}{2} \quad . \tag{9}$$

What is the ”signal”, the ”noise” and the ”signal-to-noise ratio”, for a coherent state?

4 In the semi-classical approximation, spontaneous emission is not inherently included in the model. The optical Bloch equations are derived from the equations of motion for the coefficients $c_j(t)$ that define the atomic wave function in the Schrödinger representation. For a two level atom, $c_1(t)$ and $c_2(t)$ are obviously defined through:

$$\Psi(\mathbf{r},t) = c_1(t)\Psi_1(\mathbf{r},t) + c_2(t)\Psi_2(\mathbf{r},t) \quad . \tag{10}$$

The equations of motion for $c_1(t)$ and $c_2(t)$ are

$$\mathrm{i}\frac{\mathrm{d}c_1}{\mathrm{d}t} = \Omega\cos\omega t \exp\left(-\mathrm{i}\omega_0 t\right)c_2(t) \tag{11}$$

$$\mathrm{i}\frac{\mathrm{d}c_2}{\mathrm{d}t} = \Omega\cos\omega t \exp\left(\mathrm{i}\omega_0 t\right)c_1(t) \quad , \tag{12}$$

where Ω is the Rabi frequency, ω_0 is the atomic resonance frequency, and ω is the frequency of the light. From this, one can derive the optical Bloch equations in the rotating wave approximation:

$$\frac{\mathrm{d}\tilde{\rho}_{22}}{\mathrm{d}t} = -\frac{\mathrm{d}\tilde{\rho}_{11}}{\mathrm{d}t} = -\frac{\mathrm{i}}{2}\Omega(\tilde{\rho}_{12} - \tilde{\rho}_{21}) \tag{13}$$

$$\frac{\mathrm{d}\tilde{\rho}_{12}}{\mathrm{d}t} = \frac{\mathrm{d}\tilde{\rho}_{21}^*}{\mathrm{d}t} = \frac{\mathrm{i}}{2}\Omega(\tilde{\rho}_{11} - \tilde{\rho}_{22}) + \mathrm{i}(\omega_0 - \omega)\tilde{\rho}_{12} \quad , \tag{14}$$

where $\rho_{ij} = c_i c_j^*$ and

$$\begin{aligned} \tilde{\rho}_{12} &= \exp\left\{\mathrm{i}(\omega_0 - \omega)t\right\}\rho_{12} \\ \tilde{\rho}_{21} &= \exp\left\{-\mathrm{i}(\omega_0 - \omega)t\right\}\rho_{21} \\ \tilde{\rho}_{ii} &= \rho_{ii} \quad . \end{aligned} \tag{15}$$

4a) If one wants to include spontaneous emission into the semi-classical optical Bloch equations, the deacy has to be artificially added to the equations of motion. Show how this can be done and write the optical Bloch equations with this decay included (but still in the semi-classical model). The rotating wave approximation is still allowed.

4b) Sketch a graph showing how the population in the upper state varies with time (based on the model derived in 4a). You do not have to solve the OBE's. Just outline the general behaviour and indicate roughly a relevant time scale on the time axis. Assume that the initial state is that the atom is purely in state $|1\rangle$ and the light is off. At $t = 0$, the light turns on.

5 A photon pair state is created that excites two different continuous-mode fields. These two photons are arranged such that they excite the two different input arms in a beam splitter. Thus, the input state of the beam splitter can be written as

$$|(1_1, 1_2)_\beta\rangle = \int\int \beta(t, t')\hat{a}_1^\dagger(t)\hat{a}_2^\dagger(t')\mathrm{d}t\mathrm{d}t' \quad . \tag{16}$$

The reflectance of the beam splitter is $|\mathcal{R}|^2$ and the transmittance is $|\mathcal{T}|^2$. $\mathcal{R}$ and $\mathcal{T}$ are respectively the complex reflection and transmission coefficients.

5a) Rewrite eq. (16) in a form where it contains the creation operators for the output arms instead of those for the input arms.

5b) It is a 50:50 beam splitter, which means that the reflectance and the transmittance are equal ($|\mathcal{R}|^2 = |\mathcal{T}|^2 = 1/2$). Moreover, the two states overlap perfectly in time. In other words, the joint overlap integral is unity:

$$|J|^2 = \int\int \beta^*(t, t')\beta(t', t)\mathrm{d}t\mathrm{d}t' = 1 \quad . \tag{17}$$

For this particular case, the probability is for finding one photon in each of the two output arms at a measurement is zero. Interpret this result. What does it mean, and why does this happen?