(1)

a) The equation describes the cross-section for scattering of light by an atomic system in the general case. The differential cross-section is a measure of how much light that is scattered in a certain direction in terms of how much light that is incident. In the equation, it is assumed that the atom is initially is state $|i\rangle$ and ends up in $|f\rangle$, which may be the same as $|i\rangle$. The magnitude of the differential cross-section will be strongly affected by the proximity to intermediate states $|e\rangle$, and therefore the detunings $\omega-\omega_e$ and $\omega_{sc}-\omega_e$ are very important. The equation is the general ~~one~~ equation for scattering and it is not restricted to particular cases.

b) The incident light is visible.
All atomic resonances are in the UV or higher

$$\Rightarrow \omega_e \gg \omega, \omega_{sc}$$

We consider only the Rayleigh-scattered part of the spectrum

$$\Rightarrow \begin{cases} |f\rangle = |i\rangle \\ \omega_{sc} = \omega \\ \omega_f = \omega_i = 0 \end{cases}$$

So,

$$\frac{d\sigma(\omega)}{d\Omega} = \frac{e^4 \omega^4}{16\pi^2 \varepsilon_0^2 k^2 c^4} \left| \sum_{e} \frac{(\bar{e}_{sc}\cdot\bar{D}_{ie})(\bar{e}\cdot D_{ei}) + (\bar{e}\cdot\bar{D}_{ie})(\bar{e}_{sc}\cdot D_{ei})}{\omega_e} \right|^2$$

The important feature for the spectrum of sky light is :

$$\frac{d\sigma(\omega)}{d\Omega} \sim \omega^4$$

This means that the high frequencies will be scattered to a much higher degree. This is the reason for the "bluishness" of sky light.

(2)

The energy in a classial field:

$$\mathcal{E}_R = \sum_{\bar{k}} \sum_{p} \varepsilon_0 V \omega_k^2 \left(A_{\bar{k}p} A^*_{\bar{k}p} + A^*_{\bar{k}p} A_{\bar{k}p} \right).$$

In comparison, the hamiltonian for a QM harmonic oscillator is: (for one mode)

$$\hat{H}_{\bar{k}p} = \frac{1}{2} \hbar \omega_k \left(\hat{a}_{\bar{k}p} \hat{a}^+_{\bar{k}p} + \hat{a}^+_{\bar{k}p} \hat{a}_{\bar{k}p} \right)$$

If we quantize by comparing these two expressions, we have to do the conversions

$$\begin{cases} \sqrt{\varepsilon_0 V \omega_k^2}\, A_{\bar{k}p} \rightarrow \sqrt{\frac{1}{2} \hbar \omega_k}\, \hat{a}_{\bar{k}p} \\ \sqrt{\varepsilon_0 V \omega_k^2}\, A^*_{\bar{k}p} \rightarrow \sqrt{\frac{1}{2} \hbar \omega_k}\, \hat{a}^+_{\bar{k}p} \end{cases}$$

$$\Rightarrow \begin{cases} A_{\bar{k}p} \rightarrow \sqrt{\dfrac{\hbar}{2\varepsilon_0 \omega_k V}}\, \hat{a}_{\bar{k}p} \\ A^*_{\bar{k}p} \rightarrow \sqrt{\dfrac{\hbar}{2\varepsilon_0 \omega_k V}}\, \hat{a}^+_{\bar{k}p} \end{cases}$$

The classial vector potential modulus is (for one mode)

$$A_{\bar{k}p}(\bar{r},t) = A_{\bar{k}p} e^{-i\omega_k t + i\bar{k}\cdot\bar{r}} + A^*_{\bar{k}p} e^{i\omega_k t - i\bar{k}\cdot\bar{r}}$$

The corresponding QM operator is then:

$$\hat{A}_{kp}(\bar{r},t) = \sqrt{\frac{\hbar}{2\varepsilon_0 \omega_k V}} \left(\hat{a}_{kp} e^{-i\omega_k t + i\bar{k}\cdot\bar{r}} + \hat{a}^{+}_{kp} e^{i\omega_k t - i\bar{k}\cdot\bar{r}} \right)$$

The total vector potential operator is:

$$\hat{\bar{A}} = \sum_{\bar{k}} \sum_{p=1,2} \bar{e}_{\bar{k}p} \hat{A}_{\bar{k}p}(\bar{r},t)$$

The transversal electric field is

$$\bar{E}_T = \frac{\partial \bar{A}}{\partial t}$$

$$\begin{cases} \frac{d}{dt}\left(e^{-i\omega_k t}\right) = -i\omega_k e^{-i\omega_k t} = \omega_k e^{-i\omega_k t} \cdot e^{-i\frac{\pi}{2}} = \omega_k e^{-i(\omega_k t + \frac{\pi}{2})} \\ \frac{d}{dt}\left(e^{i\omega_k t}\right) = \omega_k e^{i(\omega_k t + \frac{\pi}{2})} \end{cases}$$

$\Rightarrow$

$$\hat{\bar{E}}(\bar{r},t) = \sum_{\bar{k}} \sum_{p=1,2} \bar{e}_{\bar{k}p} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}} \cdot$$

$$\left(\hat{a}_{kp} e^{-i\omega_k t + i\bar{k}\cdot\bar{r} - i\frac{\pi}{2}} + \hat{a}^{+}_{kp} e^{i\omega_k t - i\bar{k}\cdot\bar{r} + i\frac{\pi}{2}} \right)$$

(3)

A coherent state:

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

, where α is a complex number.

a) Expec. value for the number of photons:

$$\langle n\rangle = \langle\alpha|\hat{n}|\alpha\rangle = \langle\alpha|\hat{a}^{+}\hat{a}|\alpha\rangle$$

$$\begin{cases} \hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle \\ \langle n|\hat{a}^{+} = (\hat{a}|\alpha\rangle)^* = (\sqrt{n}\,|n-1\rangle)^* = \sqrt{n}\,\langle n-1| \end{cases}$$

Also: $\hat{a}|0\rangle = \langle 0|\hat{a}^{+} = 0$

$$\hat{a}|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \sqrt{n}\,|n-1\rangle =$$

$$= e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0} \alpha^{n+1} \sqrt{\frac{n+1}{(n+1)!}}\,|n\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \cdot \alpha|n\rangle =$$

$$= \alpha|\alpha\rangle$$

$$\langle\alpha|\hat{a}^{+} = (\hat{a}|\alpha\rangle)^* = \langle\alpha|\alpha^*$$

$$\langle\alpha|\alpha\rangle = e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^{*n}\alpha^n}{n!} = e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} =$$

$$= e^{-|\alpha|^2} \cdot e^{|\alpha|^2} = 1$$

$$\left\{ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right\}$$

Thus,

$\langle n \rangle = \alpha^* \alpha = |\alpha|^2$

Uncertainty: $(\Delta n)^2 = \langle n^2 \rangle - \langle n \rangle^2$

$\langle n^2 \rangle = \langle \alpha | \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} | \alpha \rangle$

$[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger \hat{a} = 1$

$\Rightarrow \hat{a}\hat{a}^\dagger = \hat{a}^\dagger \hat{a} + 1$

$\langle n^2 \rangle = \langle \alpha | \hat{a}^\dagger (\hat{a}^\dagger \hat{a} + 1) \hat{a} | \alpha \rangle = \alpha^* \alpha^* \alpha \alpha + \alpha^* \alpha =$
$= |\alpha|^4 + |\alpha|^2$

$\Rightarrow (\Delta n)^2 = |\alpha|^4 + |\alpha|^2 - |\alpha|^2 = |\alpha|^4$

$\Delta n = |\alpha|^2 = \langle n \rangle$

b) $\hat{E}(x) = \frac{1}{2} \hat{a} e^{-ix} + \frac{1}{2} \hat{a}^\dagger e^{ix}$, $x = \omega t - kz - \frac{\pi}{2}$

Signal: $S = \langle \hat{E} \rangle = \langle \alpha | \hat{E} | \alpha \rangle =$

$= \frac{1}{2} e^{-ix} \alpha + \frac{1}{2} e^{ix} \alpha^*$

Let $\alpha = |\alpha| e^{i\theta}$

$\Rightarrow S = |\alpha| \frac{1}{2} \left(e^{i(\theta - x)} + e^{-i(\theta - x)} \right) = |\alpha| \cos(\theta - x)$

Noise: $N = (\Delta E(x))^2 = \langle \hat{E}^2 \rangle - \langle \hat{E} \rangle^2$

$$\hat{E}^2 = \frac{1}{4} \hat{a} \hat{a} e^{-2ix} + \frac{1}{4} \hat{a}^\dagger \hat{a}^\dagger e^{2ix} + \frac{1}{4} \hat{a} \hat{a}^\dagger + \frac{1}{4} \hat{a}^\dagger \hat{a}$$

$$\langle \hat{E}^2 \rangle = \frac{1}{4} \alpha \alpha e^{-2ix} + \frac{1}{4} \alpha^* \alpha^* e^{2ix} + \frac{1}{4} \left(|\alpha|^2 + 1 \right) + \frac{1}{4} |\alpha|^2 =$$

$$= \frac{1}{4} + \frac{1}{2} |\alpha|^2 + \frac{1}{4} |\alpha|^2 e^{2i(\theta - x)} + \frac{1}{4} |\alpha| e^{-2i(\theta - x)}$$

$$\langle \hat{E} \rangle^2 = \frac{1}{4} |\alpha|^2 \left(e^{2i(\theta - x)} + e^{-2i(\theta - x)} + 1 + 1 \right)$$

$$\Rightarrow \cancel{\Delta E} \; N = \frac{1}{4}$$

Signal to noise:

$$SNR = \frac{S^2}{N} = 4 |\alpha|^2 \cos^2(\theta - x)$$

(4)

a) An extra decay from state $|2\rangle$ is added. Without any coupling ($\Omega = 0$) we would have

$$\frac{dc_2}{dt} = -\gamma c_2 \Rightarrow c_2(t) = c(0)\, e^{-\gamma t}$$

The population: $|c_2(t)|^2 = |c_2(0)|^2 e^{-2\gamma t}$

With coupling, we simply add this term to the eq. of motion for c_2

$$\Rightarrow i\frac{dc_2}{dt} = \Omega \cos\omega t\, e^{i\omega_0 t} c_1 - i\gamma c_2$$

$$\frac{dc_2}{dt} = -i\Omega\cos\omega t\, e^{i\omega_0 t} c_1 - \gamma c_2$$

$$\frac{dc_1}{dt} = -i\Omega\cos\omega t\, e^{-i\omega_0 t} c_2$$

The OBE:

$$\frac{d\rho_{22}}{dt} = \frac{d}{dt}(c_2^* c_2) = c_2^*\frac{dc_2}{dt} + \frac{dc_2^*}{dt}c_2 =$$

$$= -i\Omega\cos\omega t\, e^{i\omega_0 t}\rho_{12} + i\Omega\cos\omega t\, e^{-i\omega_0 t}\rho_{21} - \gamma\rho_{22} - \gamma\rho_{22} -$$

$$= -\frac{i}{2}\Omega\left(e^{i\omega t} + e^{-i\omega t}\right)\left(e^{i\omega_0 t}\rho_{12} - e^{-i\omega_0 t}\rho_{21}\right) - 2\gamma\rho_{22}$$

RWA!

$$\Rightarrow \frac{d\rho_{22}}{dt} = -\frac{i}{2}\Omega\left(-\tilde{\rho}_{21} + \tilde{\rho}_{12}\right) - 2\gamma\rho_{22} = -\frac{d\rho_{11}}{dt}$$

$$\frac{d\rho_{12}}{dt} = \frac{d}{dt}\left(c_1 c_2^*\right) = c_1 \frac{dc_2^*}{dt} + \frac{dc_1}{dt} c_2^* =$$

$$= i\Omega \cos\omega t\, e^{-i\omega_0 t} \rho_{11} - \gamma \rho_{12} - i\Omega \cos\omega t\, e^{-i\omega_0 t} \rho_{22} =$$

$$= \frac{i}{2}\Omega \left(e^{i\omega t} + e^{-i\omega t}\right) e^{-i\omega_0 t} \left(\rho_{11} - \rho_{22}\right) - \gamma \rho_{12}$$

RWA!

$$\frac{d\rho_{12}}{dt} = \frac{i}{2}\Omega\, e^{i(\omega-\omega_0)t} \left(\rho_{11} - \rho_{22}\right) - \gamma \rho_{12}$$

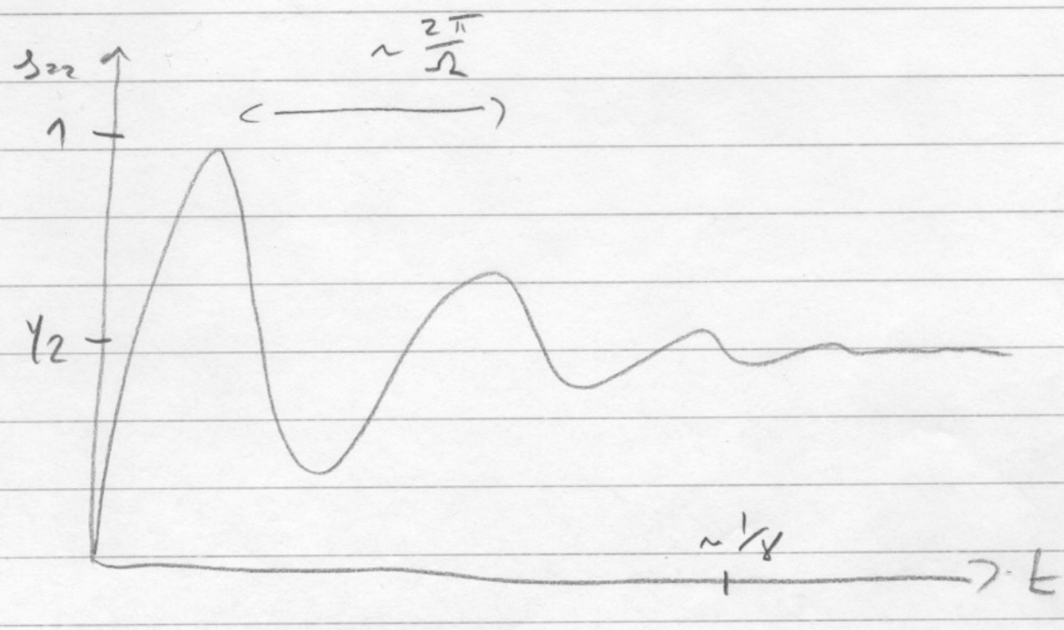
$$\hat{\rho}_{12} = e^{i(\omega_0-\omega)t} \rho_{12}$$

$$\Rightarrow \frac{d\hat{\rho}_{12}}{dt} = i(\omega_0-\omega)\tilde{\rho}_{12} + e^{i(\omega_0-\omega)t} \frac{d\rho_{12}}{dt} =$$

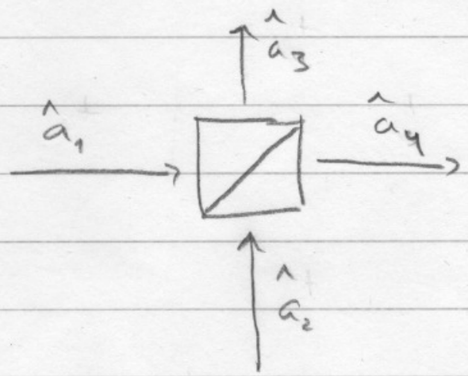
$$= i(\omega_0-\omega)\tilde{\rho}_{12} + \frac{i}{2}\Omega\left(\rho_{11} - \rho_{22}\right) - \gamma \hat{\rho}_{12} =$$

$$= \frac{i}{2}\Omega\left(\rho_{11} - \rho_{22}\right) - \left\{ i(\omega_0-\omega) - \gamma \right\} \hat{\rho}_{12}$$

b)



5 ~~4~~



$$\begin{cases} \hat{a}_3 = R\hat{a}_1 + T\hat{a}_2 \\ \hat{a}_4 = T\hat{a}_1 + R\hat{a}_2 \end{cases}$$

$$\begin{cases} |R|^2 + |T|^2 = 1 \\ RT^* + T^* R = 0 \end{cases}$$

$$\Rightarrow \begin{cases} R^*\hat{a}_3 = |R|^2\hat{a}_1 + R^* T\hat{a}_2 \\ T^*\hat{a}_4 = |T|^2\hat{a}_1 + T^* R\hat{a}_2 \end{cases}$$

$$\Rightarrow \hat{a}_1 = R^*\hat{a}_3 + T^*\hat{a}_4$$

$$\hat{a}_2 = T^*\hat{a}_3 + R^*\hat{a}_4 \quad \text{(in the same way)}$$

$$\Rightarrow \begin{cases} \hat{a}_1^+ = R\hat{a}_3^+ + T\hat{a}_4^+ \\ \hat{a}_2^+ = T\hat{a}_3^+ + R\hat{a}_4^+ \end{cases}$$

input state:

$$|(1_1, 1_1)_\beta\rangle = \iint \beta(t,t')\,\hat{a}_1^+(t)\,\hat{a}_2^+(t')\,dt\,dt'\,|0\rangle$$

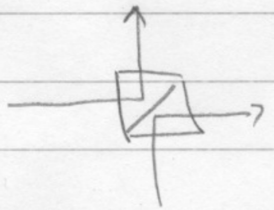
$$\hat{a}_1^+(t)\,\hat{a}_2^+(t') = RT\,\hat{a}_3^+(t)\,\hat{a}_3^+(t') + TR\,\hat{a}_4^+(t)\,\hat{a}_4^+(t') +$$
$$+ R^2\,\hat{a}_3^+(t)\,\hat{a}_4^+(t') + T^2\,\hat{a}_4^+(t)\,\hat{a}_3^+(t')$$

$\Rightarrow \quad |(1_1, 1_2)_\beta\rangle =$

$$= \iint \beta(t,t') \Big\{ RT\, \hat{a}_3^+(t)\, \hat{a}_3^+(t') + TR\, \hat{a}_4^+(t)\, \hat{a}_4(t') + R^2\, \hat{a}_3^+(t)\, \hat{a}_4^+(t') + T^2\, \hat{a}_4^+(t)\, \hat{a}_3^+(t') \Big\}\, dt\, dt' \;|0\rangle$$

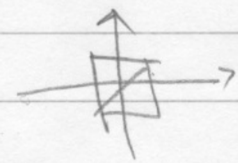
b) This happens because there are two ways to create the state $|(1_3, 1_4)_\beta\rangle$.

Namely:



$\sim \cancel{R\hat{a}_1^+}\; R^2\, \hat{a}_1^+(t)\, \hat{a}_2^+(t')$

and:



$\sim T^2\, \hat{a}_1^+(t)\, \hat{a}_2^+$

Since $|R|^2 = |T|^2 = \frac{1}{2}$, the probability amplitude for these two occurences are the same. However, they are out of ~~phase~~ phase.
Thus, the total probability cancels.