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110530

Examination in **Statistical Physics II**, May 30, 2011.

Allowed accessories: *Physics Handbook*, calculator.

Define all notation. State clearly assumptions and approximations.

3 problems on 2 pages, 4 points each, maximally 12 points. With hand-in problems, maximally 24 points. G: 12p, VG: 18p.

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1. *True or false?* For each of the items a-h below, is it true or false?

   (a) The ferromagnetic phase transition in the Ising model is due to quantum effects.

   (b) Mean field theory gives the right critical exponents in high enough dimensions.

   (c) According to universality, the Ising model and XY model must have the same critical exponents.

   (d) Near a critical point, renormalization flow must go towards the critical point from all directions.

   (e) The superfluid part of moving ^{4}He carries entropy but no energy.

   (f) High-temperature series expansion gives the same critical exponents as mean-field theory.

   (g) A Maxwell construction is used to find the critical temperature for a liquid-gas transition.

   (h) The Landau-Ginzburg equation for a superconductor holds only close to the critical temperature.

   (**Grading** of problem 1: Each correct answer gives 0.5 points, each incorrect answer gives -0.5 points.)

   (a) False: The Ising model we studied is classical and the phase transition arises due to classical physics.

   (b) True

   (c) False: According to universality, two models have the same critical exponents if they have the same dimension and same symmetry of the order parameter.

   (d) False: The flow is usually away from the critical point in at least one direction.

(e) *False: It carries no entropy.*

(f) *False: High-temperature expansions converge towards the true critical exponents.*

(g) *False: It is used to find the pressure and volume at the phase transition at temperatures below* T_c*. It cannot find* T_c *itself.*

(h) *True.*

2. *Renormalization group.* Consider the Ising model in one dimension with periodic boundary conditions,

$$\tilde{H} = -J\sum_{i=1}^{N}\sigma_i\sigma_{i+1} - \tilde{h}\sum_{i=1}^{N}\sigma_i$$

with $\sigma_i = \pm 1$ and $\sigma_{N+1} = \sigma_1$. For convenience, define the dimensionless Hamiltonian

$$H = -\beta\tilde{H} = K\sum_{i=1}^{N}\sigma_i\sigma_{i+1} + h\sum_{i=1}^{N}\sigma_i,$$

where $K = \beta J$, $h = \beta\tilde{h}$.

(a) Write down the partition function and carry out the renormalization group transformation to obtain the renormalized Hamiltonian. **Given:** You can use the formulas

$$2e^{q(\sigma_a+\sigma_b)/2}\cosh\left[p(\sigma_a+\sigma_b)+q\right] = \exp\left\{2g + x\sigma_a\sigma_b + \frac{1}{2}y(\sigma_a+\sigma_b)\right\},$$

where

$$x = \frac{1}{4}\ln\frac{\cosh(2p+q)\cosh(2p-q)}{\cosh^2 q},$$

and

$$y = q + \frac{1}{2}\ln\frac{\cosh(2p+q)}{\cosh(2p-q)},$$

and g is also a complicated function of p and q which you don't have to worry about. Recall the definition

$$\cosh z = \frac{e^z + e^{-z}}{2}.$$

(b) It is possible to find from the above (but you don't have to prove it), that $K = 0$ and $K = \infty$ are fixed points. What does it mean mathematically and physically?

(a) *See book.*

(b) Mathematically, $K' = K$ for all coupling constants K. Physically, invariance under change of scale means correlation length is either zero or infinite, the latter means critical point.

3. *Leggett's derivation of the Gross-Pitaevskii equation.* A system of N bosons is described by the many-body wavefunction

$$\psi(r_1, \ldots r_N, t).$$

(a) Write down the Ansatz for this wavefunction that is supposed to describe a Bose-Einstein condensed gas.

(b) Inserting the Ansatz into the many-body Schrödinger equation, one ends up with an equation of motion

$$i\hbar N \int dr \varphi^*(r) \frac{\partial}{\partial t} \varphi(r) = N \int dr \varphi^*(r) \left(\frac{\hbar^2}{2m} \nabla^2 + V(r) \right) \varphi(r) + N(N-1) \frac{U_0}{2} \int dr |\varphi(r)|^4.$$

You do not have to do the steps leading up to the equation above. But from this, I ask you to derive the Gross-Pitaevskii equation and prove that the eigenvalue of the Gross-Pitaevskii equation is the chemical potential of the system.

Solution: In my lecture notes.