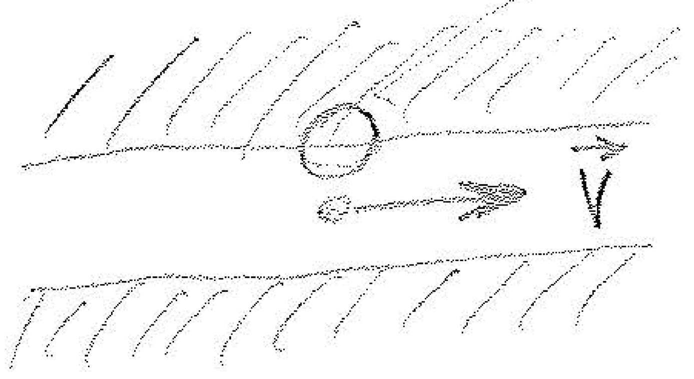
11.2.1 SF: qualitative features.

Superfluidity

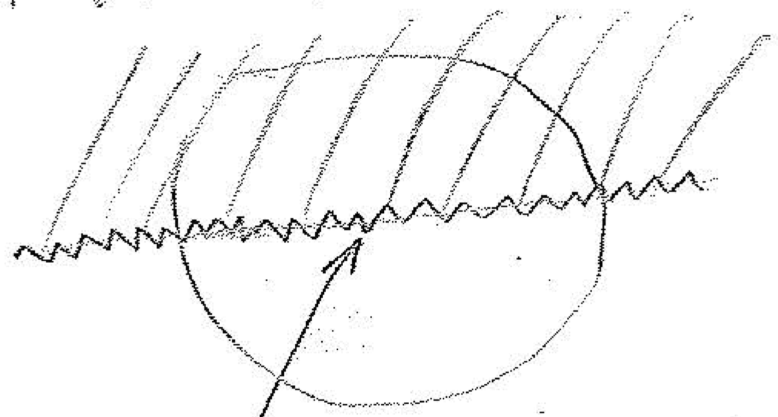
Dmitry Kobyakov 2011

SF – name because of its property:



no friction

Friction:

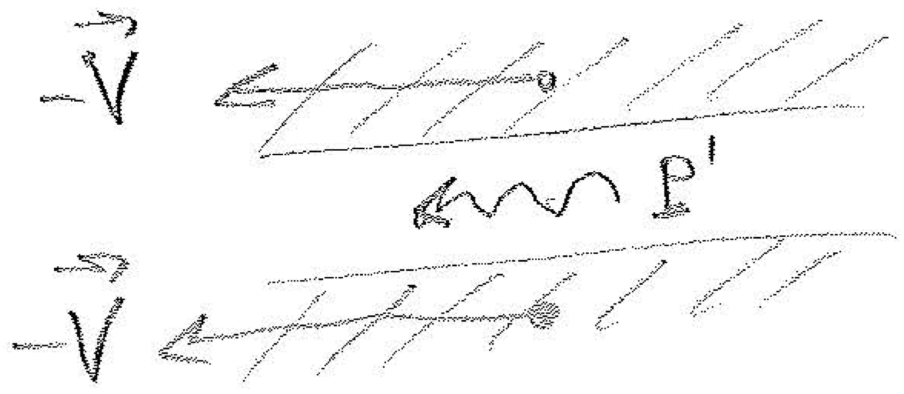


interaction → momentum exchange → dissipation.

Why this happens?

Consider excitation spectrum of (quasi)particles.

① SF-rest frame of reference (X')



Creates 1 excitation $\epsilon(p) \Rightarrow E_0' = \epsilon(p)$, $p' = p$.

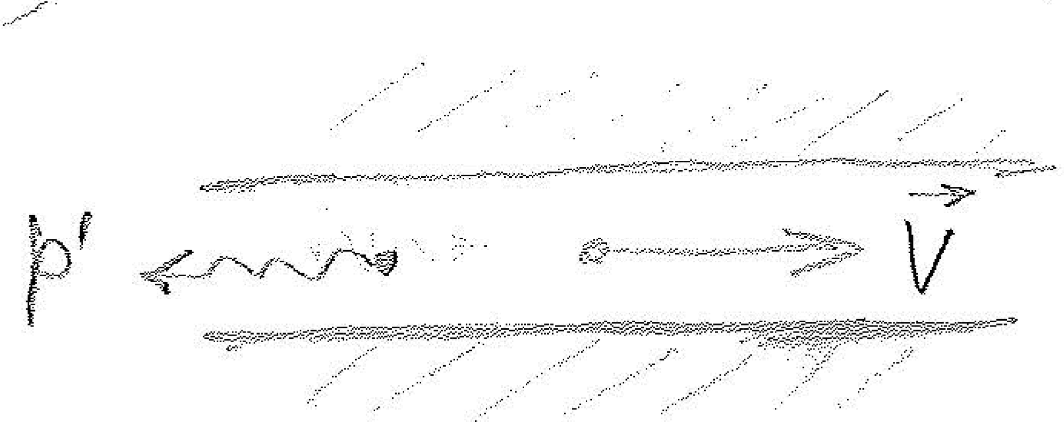
Energy of the whole condensate

No excitations: $E_0' = 0$

$$\boxed{E_0' = \frac{p'^2}{2M}}$$

② Container (lab)-rest fr.-of-ref. (X)

$$\boxed{\vec{X} = \vec{X}' + \vec{V}_0 t}$$



Transformation of energy & momentum of the whole condensate:

1 excitation: $E_0 = \frac{MV^2}{2} + \epsilon(p) + \vec{p}\vec{V}$.

No excitations: $E_0 = \frac{M\vec{V}^2}{2}$

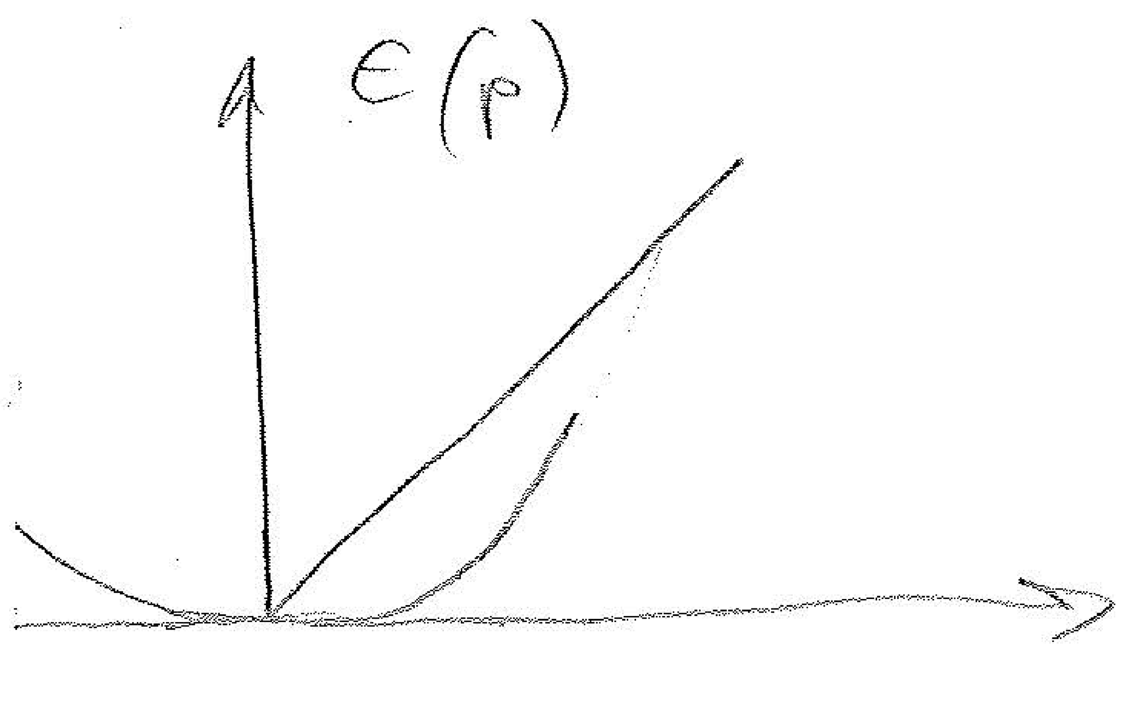
$$\Rightarrow \boxed{\epsilon(p) + \vec{p}\vec{V} < 0}$$

Reminder:

$E = \frac{\vec{P}^2}{2M} = \frac{M\vec{V}^2}{2}$

$$\vec{P} \to \vec{P} + \vec{P}_1 \Rightarrow \vec{V} \to \vec{V} + \frac{\vec{P}_1}{M} \Rightarrow E \to \frac{M\left(\vec{V} + \frac{\vec{P}_1}{M}\right)^2}{2} = \overset{\frac{MV^2}{2}}{E} + \overset{E_0'}{\frac{\vec{P}_1^2}{2M}} + \vec{P}_1\vec{V}.$$

Physical meaning.



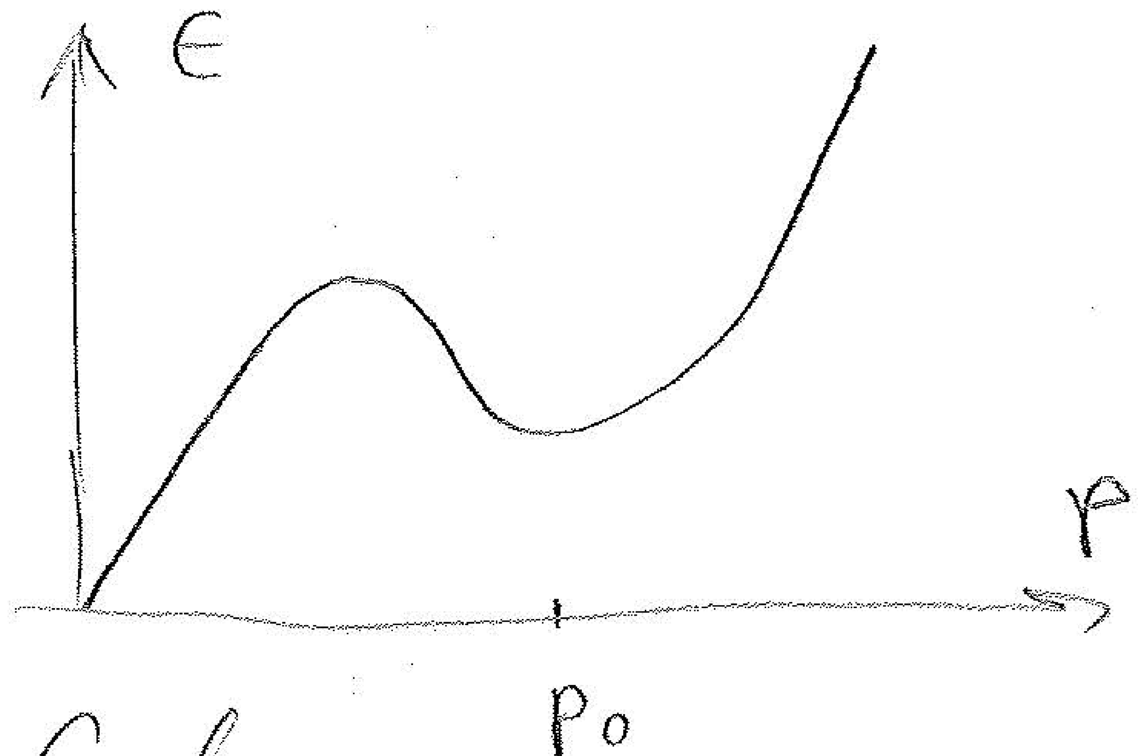
$$\frac{\epsilon(p)}{p} = \tan\{\epsilon(p) \text{ slope}\}$$

Quadratic: $\lim_{p\to 0} \frac{\epsilon(p)}{p} = 0$.

Linear: $\lim_{p\to 0} \frac{\epsilon(p)}{p} = C_s$,

where $\epsilon(p) = C_s p$, $C_s = \text{const}$.

Real superfluid: low-energy (inelastic) n-scattering



$$\vec{V} = \frac{\partial \epsilon}{\partial \vec{p}} \quad \text{- quasiparticle velocity}$$

$$\vec{V}(\vec{p}_0) = 0$$

Conclusion: non-interacting particles are not SF.

p close to p_0: rotons.

SF velocity

Condensate = coherent macroscopic quantum system described by order-parameter (or w/f) Ψ.

Reminder:

$$i\hbar \partial_t \psi = -\frac{\hbar^2}{2m}\Delta\psi, \qquad \psi^*\psi = n.$$

$$\Downarrow$$

$$\begin{cases} \partial_t \psi = i\frac{\hbar}{2m}\Delta\psi. \\ \partial_t \psi^* = -i\frac{\hbar}{2m}\Delta\psi^* \end{cases}$$

$$\partial_t n = (\partial_t\psi^*)\psi + \psi^*\partial_t\psi = \frac{i\hbar}{2m}\left(\psi^*\Delta\psi - \psi\Delta\psi^*\right).$$

Fundamental equation: $\boxed{\partial_t n + \operatorname{div}(n\vec{V}) = 0} \iff \partial_t n + \nabla(n\vec{V}) = 0.$

Easily: $\psi^*\Delta\psi - \psi\Delta\psi^* = \nabla(\psi^*\nabla\psi - \psi\nabla\psi^*)$.

$$\partial_t n = \frac{i\hbar}{2m}\nabla\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) \Rightarrow \vec{j} = n\vec{V} = \frac{i\hbar}{2m}\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right). \tag{11.33}$$

$\psi = \sqrt{n}\, e^{i\varphi} = a(r)\, e^{i\varphi}$

$j_s = \hbar n \nabla\varphi = m n \frac{\hbar}{m} \nabla\varphi$, $[n] = \frac{\text{particles}}{m^3}$, $[j] = \frac{kg}{m^3} \cdot \frac{m}{s}$.

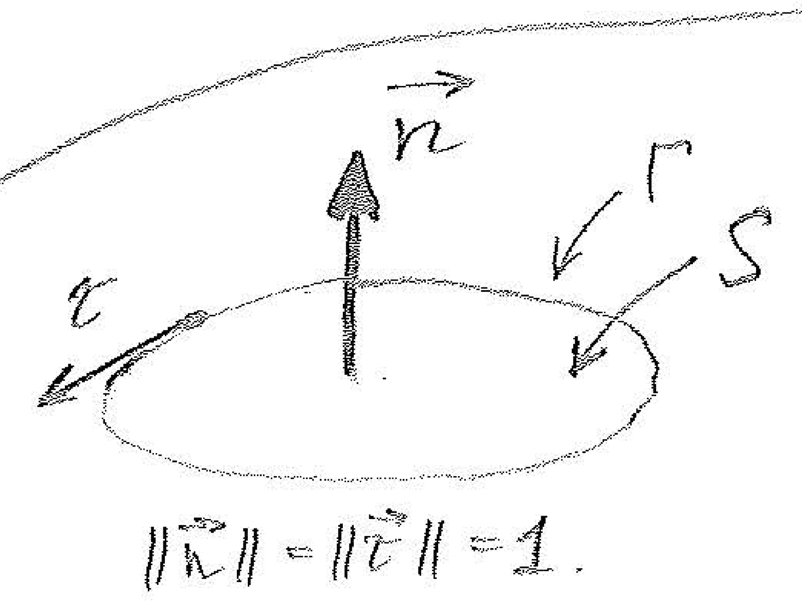
Velocity $= \frac{\hbar}{m}\nabla\varphi$. Check: $\left[\frac{\hbar}{m}\nabla\varphi\right] = \frac{J\cdot s}{kg}\,\frac{1}{m} = \frac{kg\,\frac{m^2}{s^2}\cdot s}{kg\cdot m} = \frac{m}{s}$

$\Downarrow$

$\text{rot}\{\text{Velocity}\} = 0$ (irrotational)

Reminder

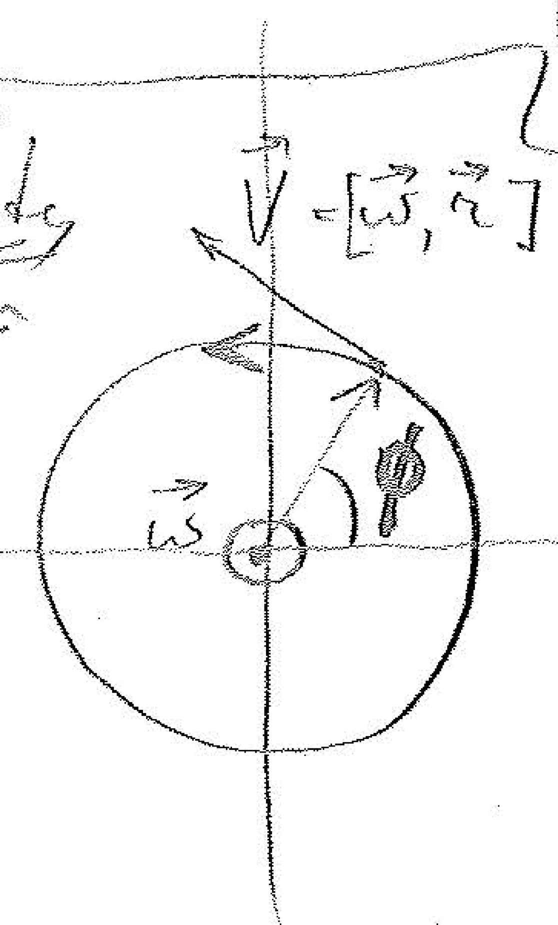
Vorticity of a field $\vec{A}$: $\text{rot}\vec{A}$.



Meaning: $$\boxed{\iint_S \vec{n}\, dS \cdot \text{rot}\vec{A} = \oint_\Gamma \vec{\tau}\, dl \cdot \vec{A}}$$

Green-Ostrogradsky

Rigid-body rotation



$$\oint d\vec{l}\,\vec{V} = \int_0^{2\pi} r\, d\phi\, \omega r = 2\pi \underbrace{r^2}_{\text{area}} \omega \Rightarrow \text{rot}\vec{V} = 2\vec{\omega}.$$

Other way to prove:

$$\text{rot}\vec{V} = [\nabla, [\vec{\omega}, \vec{r}]] = \vec{\omega}(\nabla, \vec{r}) - \vec{r}(\vec{\nabla}, \vec{\omega}) =$$

a b c b a c c a b

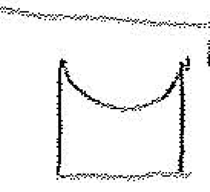
$$= \vec{\omega}(\partial_x x + \partial_y y) = 2\vec{\omega}$$

$$\iint d\vec{s}\,\text{rot}\vec{V} = \iint ds\, 2\omega = 2\pi r^2 \omega.$$

Conclusion: at regular points of Ψ the SF motion is irrotational.

Quantization of rotational SF-motion.

Also experimental motivation:



Form such that the entire liquid rotates.

Assumption: irrotationality $\text{rot}\vec{V}_{SF} = 0$ is violated at singular points.

Must be: single valued observables $\Longrightarrow$

ψ can change its phase only by $2\pi l_0$ on a closed loop, $l_0 \in \mathbb{Z}$.

Phase is therefore: $\varphi = \varphi_0 + l_0 \phi$, ϕ - angle.

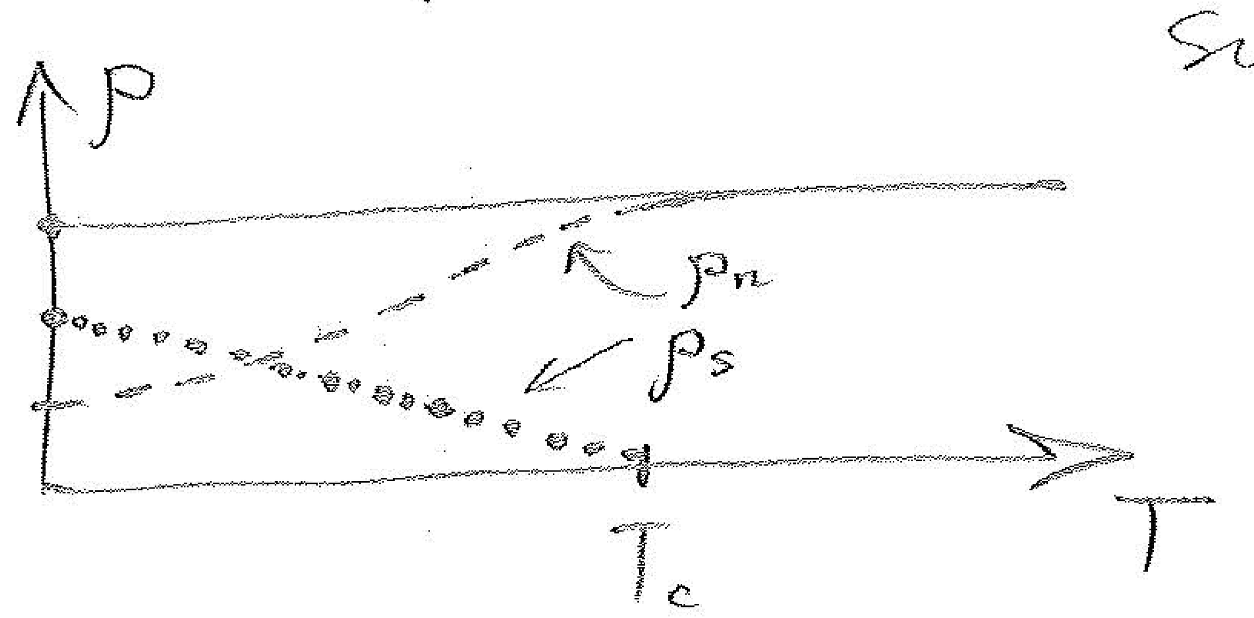
$$\nabla\varphi = \frac{1}{r}\vec{e}_\varphi \partial_\phi \left[\varphi_0 + l_0\phi\right] = \frac{l_0}{r}\vec{e}_\varphi \qquad \vec{V}_{SF} = \frac{l_0}{r}\vec{e}_\phi, \quad \|\vec{e}_\phi\| = 1$$

Thus,

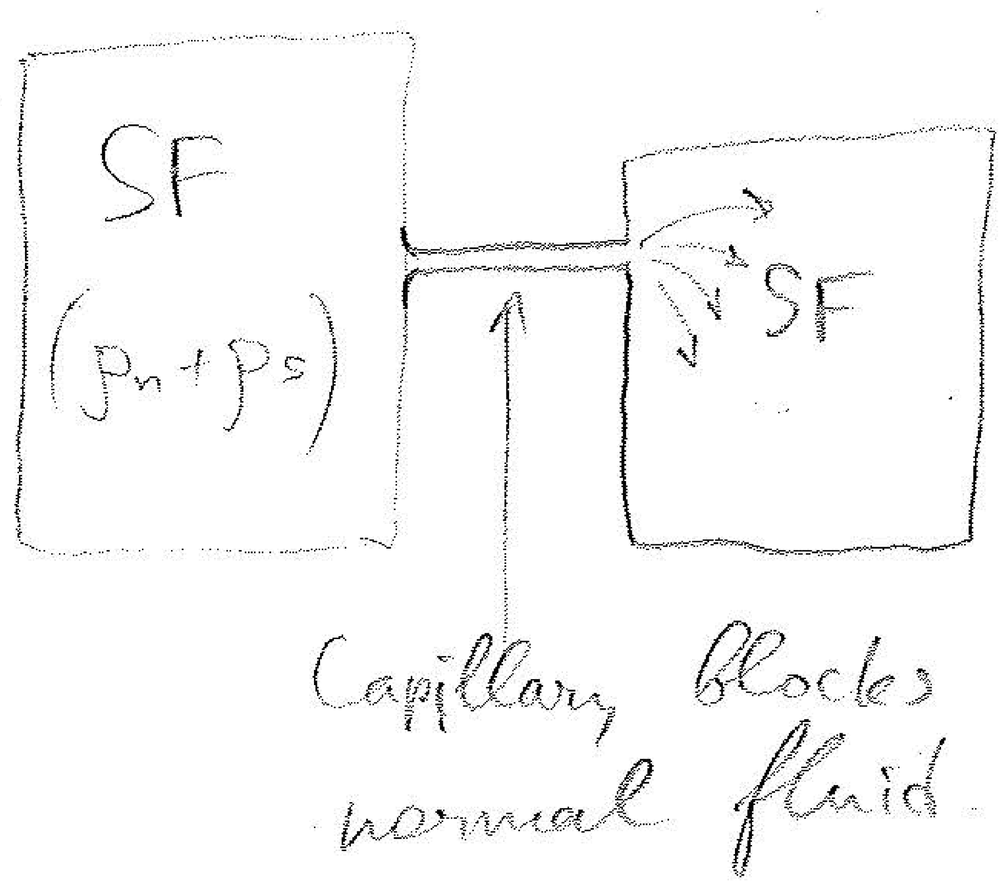
$$\oint \vec{V}_{SF}\, d\vec{l} = \oint r\, d\phi \frac{\hbar}{m}\frac{l_0}{r} = \underbrace{2\pi l_0 \frac{\hbar}{m}}_{\text{quantization.}} \qquad (11.39).$$

„Thermo-mechanical" effect.

Real quantum fluid: Conventionally: superfluid + normal „parts".



Effect:



$$dE = TdS + \underbrace{\mu}_{0\,(SF)}\, \underbrace{dN}_{=0\,(\text{normal})} - P\underbrace{dV}_{=0}$$

$\boxed{\text{SF carries no enthropy}}$

1 gramm of SF He^4 leaked. Total entropy is not changed and redistributes to a smaller M.

$$\Downarrow$$

$\Delta Q = Ts \cdot 1\,\text{gram} > 0$ – in the left container. (s = enthropy per 1 gr.)

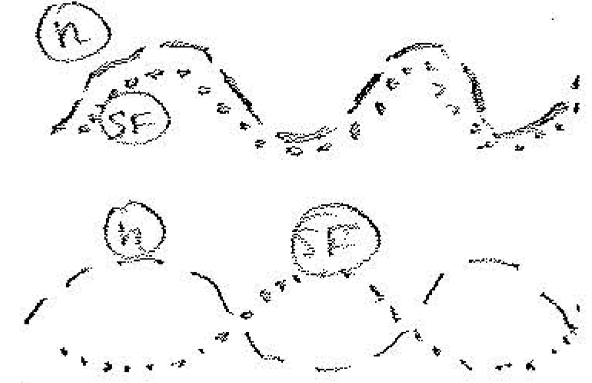
Sound in SF.

Continuity eqn + Euler eqn for small $\vec{j}$:

$$\begin{cases} \partial_t \rho + \operatorname{div}\vec{j} = 0 \\ \partial_t \vec{j} + \operatorname{grad} P = 0. \end{cases} \Rightarrow \partial^2_{tt}\rho - \nabla^2 P = 0.$$

Pressure & density are related: $dP = \left(\frac{\partial P}{\partial \rho}\right)_s d\rho$. Denote $c_s = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_s}$

Then $\boxed{\partial^2_{tt}\rho - c^2\nabla^2\rho = 0}$

Landau Lifshitz have shown: there are (c_1) first sound 
(c_2) second sound.

Landau's prediction: $c_2 = \frac{c_1}{\sqrt{3}}$. Correct for $\lim_{T\to 0}$. 5

Effect:  $\vec{B}$
Qualitatively: Lenz rule

SC : qualitative features

Idea: charged particles with attractive interaction pair: $(\uparrow \vec{p} + -\vec{p} \downarrow)$.

Reminder: Gauge-invariance of e-m field.

Vector potential $\vec{A}$: $\vec{B} = \text{rot}\,\vec{A}$. B - measurable.

Measurable not changes when $\vec{A} \to \vec{A} + \nabla\chi(r)$

Action: $S = \int d^3r\, \mathcal{L}$. (Gauge tranform.)

Lagrangian contains $\vec{A}$ as: $\frac{e}{c}\int d^3\vec{r}\, \vec{A}(\vec{r})$.

Wavefunction (Feynman): $\Psi \sim e^{\frac{i}{\hbar}S}$

macrosc. ($N \to +\infty$) ord. par. of SC: $\Xi = \langle N_{full}, N \,|\, \hat{\Psi}(x)\hat{\Psi}(x') \,|\, N_{full}, N+2 \rangle$

(not just $\langle N_{full}, N | \Psi | N_{full}, N+1\rangle$ since we have pairing)

Thus, the gauge transf. gives:

$$\Xi = \hat{\Psi}(x)\hat{\Psi}(x') \longrightarrow e^{\frac{2ie}{\hbar c}\chi(\vec{r})}\hat{\Psi}(x)\hat{\Psi}(x'),$$

i.e. phase of Ξ changes: $\varphi \to \varphi + \frac{2e}{\hbar c}\chi(\vec{r})$.

However the current of Ξ is also observ. $\Rightarrow$

$\Rightarrow$ it must be gauge-invar. $\Rightarrow$ take

instead $\vec{j}_s = \frac{\hbar}{2m} n_s \nabla\varphi$: $\vec{j}_s = -\left(\frac{e^2}{mc}\vec{A} + \frac{e\hbar}{2m}\nabla\varphi\right) n_s$

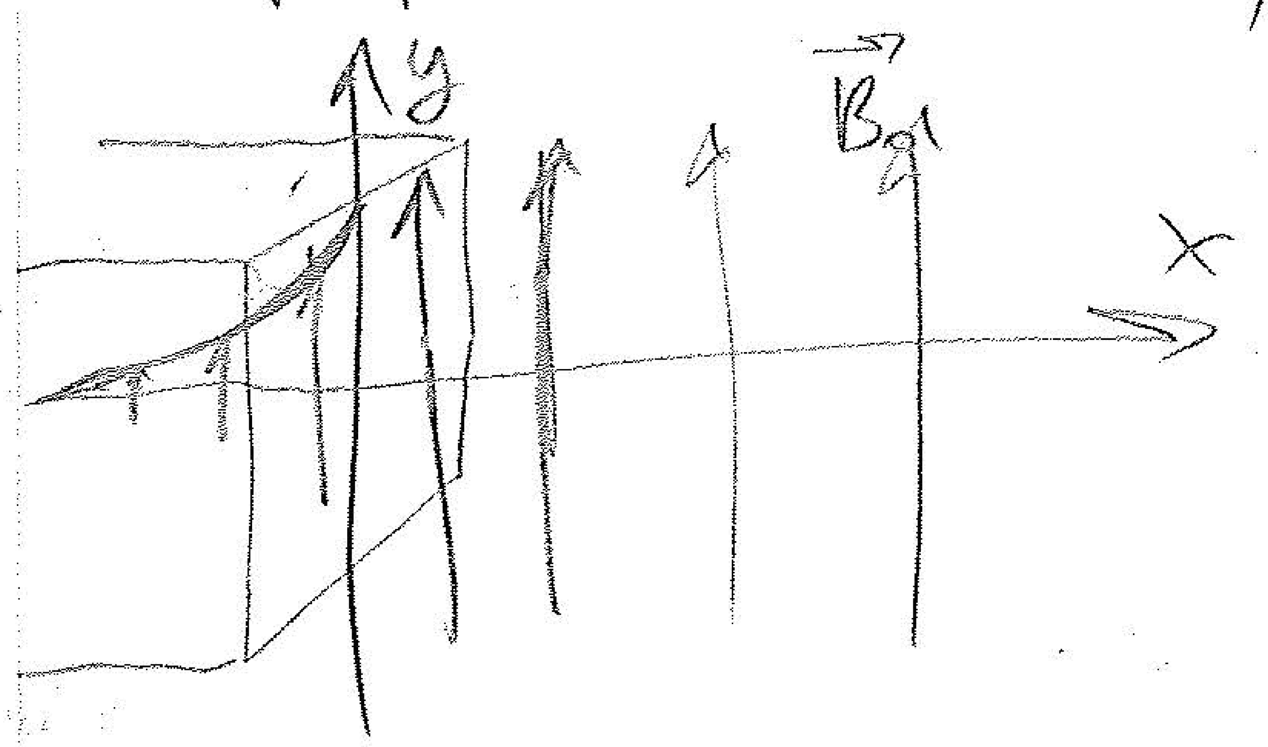
(Book has $|\psi|^2 = \frac{n_s}{2}$, where n_s - density of SC electrons)

Assume that $j_{normal} = 0$. Take rot from $\vec{j}_s$:

$$\boxed{\mathrm{rot}\, \vec{j}_s = -\frac{e^2 n_s}{mc} \vec{B}}$$ Specific for SC. (Londons eqn.)

Maxwell eqns (2 of them) $\begin{cases} \mathrm{rot}\, \vec{B} = \frac{4\pi}{c} \vec{j} \\ \mathrm{div}\, \vec{B} = 0 \end{cases} \longrightarrow \mathrm{rot}(\mathrm{rot}\, \vec{B}) = -\nabla^2 \vec{B}$

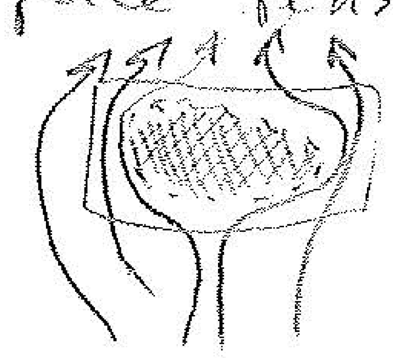
$$\nabla^2 \vec{B} = \delta^{-2} \vec{B}, \quad \delta^2 = \frac{mc^2}{4\pi e^2 n_s}, \quad \delta \sim 10^{-6} - 10^{-5}\,\mathrm{cm}.$$



$$\vec{B} \perp \vec{e}_x$$

$$\vec{B}(x) = \vec{B}_0 e^{-\frac{x}{\delta}}.$$

$\xi \gg \delta$ — type I SC. (Surface tension of SC phase > 0).



G free energy functional

Close to $T = T_c - 0$ n_s must $\longrightarrow 0$. Free energy:

$$G = G_{normal} + \int d^3 r \left[\frac{\hbar^2}{4m} |\nabla \Psi|^2 + a |\Psi|^2 + \frac{b}{2} |\Psi|^4 \right].$$

$$\frac{\delta G}{\delta \Psi^*} \longrightarrow (11.112).$$

Coherence length: $-\frac{\hbar^2}{4m} \nabla^2 \sim a \Rightarrow \xi_0 = \frac{\hbar}{\sqrt{8m|a|}}.$

So called TFA:

$G_S - G_N = aN + \frac{b}{2}N^2$, N – number of SC electrons

For a given T minimise $G_S - G_N$: $a + bN = 0 \to N = -\frac{a}{b}$,

$a = -(T_c - T)\alpha$. $N = \frac{\alpha}{b}(T_c - T)$

$$G_s - G_N = a\frac{\alpha}{b}(T_c - T) + \frac{b}{2}\frac{\alpha^2}{b^2}(T_c - T)^2 = -\frac{\alpha^2}{b}(T_c - T)^2 + \frac{\alpha^2}{2b}(T_c - T)^2 =$$
$$= -\frac{\alpha^2}{2b}(T_c - T)^2.$$

Specific heat: $C_s - C_N = V \cdot \frac{\alpha^2 T_c}{b}$.

Mag. field energy: $-\frac{H_c^2}{8\pi}V$ (note a mistake on p. 456).

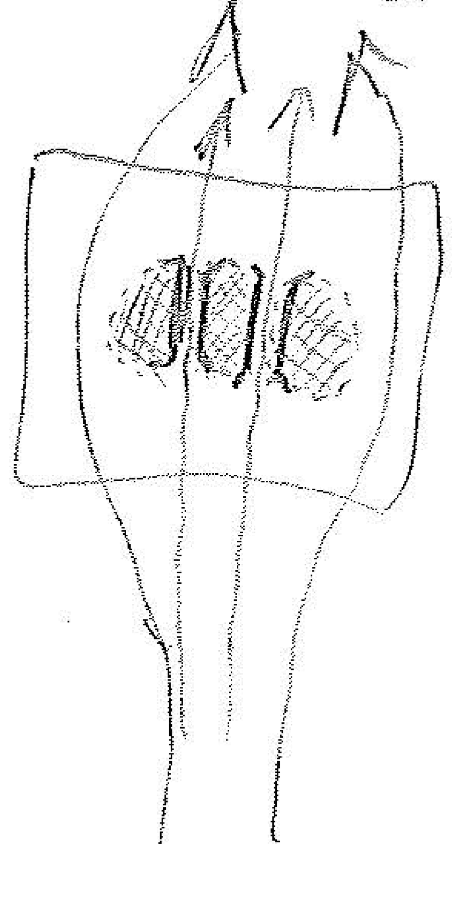
$G_s - G_N \sim E_{mag} \longrightarrow H_c = \sqrt{\frac{4\pi\alpha^2}{b}}\,(T_c - T)$. – critical field, destructing the SC.

G-L equations

Note: mag. field modifies: $|\nabla\Psi|^2 \to \left|\left(\nabla + \frac{2ie}{\hbar c}\vec{A}\right)\Psi\right|^2$.

This eqn. (11.120) may be solved for a general G-L parameter $\frac{\delta(T)}{\xi(T)}$.

For $\delta > \xi$: type II SC.



Vortices.