Characteristics of Small World Networks

Petter Holme

20th April 2001

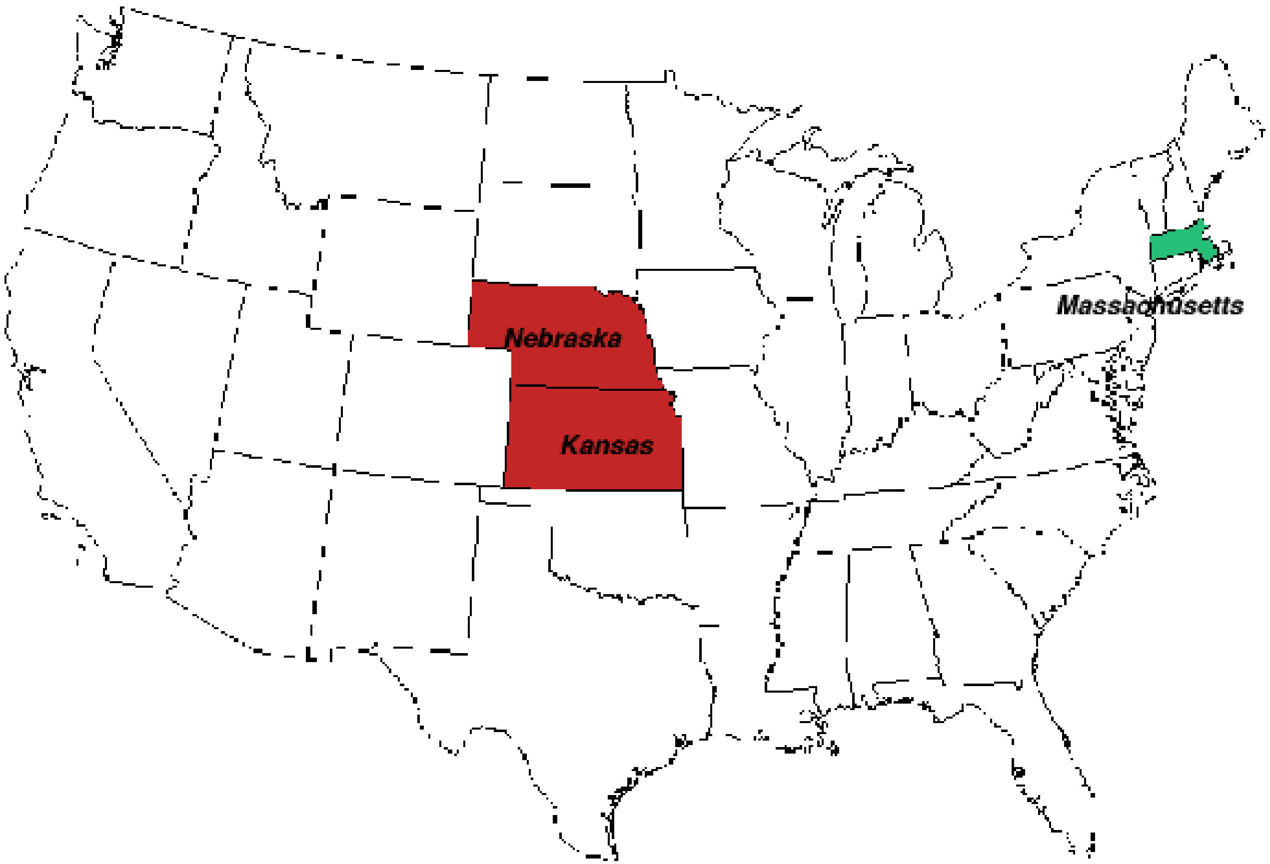
References:

[1.] D. J. Watts and S. H. Strogatz, *Collective Dynamics of 'Small-World' Networks*, Nature **393**, 440 (1998).
[2.] D. J. Watts , *Small Worlds: The Dynamics of Networks between Order and Randomness*, (Princeton University Press, Princeton, 1999), Part 1.
[3.] N. Mathias and V. Gopal, *Small Worlds: How and Why*, Phys. Rev. E **63**, 21117 (2001).
[4.] M. Gitterman, *Small-World Phenomena in Physics: The Ising Model*, J. Phys. A **33**, 8373 (2000).

Contents

Milgram's Experiment

S. Milgram, *The Small World Problem*, Psycol. Today **2**, 60 (1967).



Characteristic path length $L \approx 5$. ($\Rightarrow L = 6$ for the whole world.)

Some Graph Theoretical Definitions

Definition 1 *The connectivity of a vertex v, k_v, is the number of attached edges.*

Definition 2 *Let $d(i,j)$ be the length of the shortest path between the vertices i and j, then the characteristic path length, L, is $d(i,j)$ averaged over all $\binom{n}{2}$ pairs of vertices.*

Definition 3 *The diameter of the graph is $D = \max_{(i,j)} d(i,j)$.*

(Obviously some confusion here.)

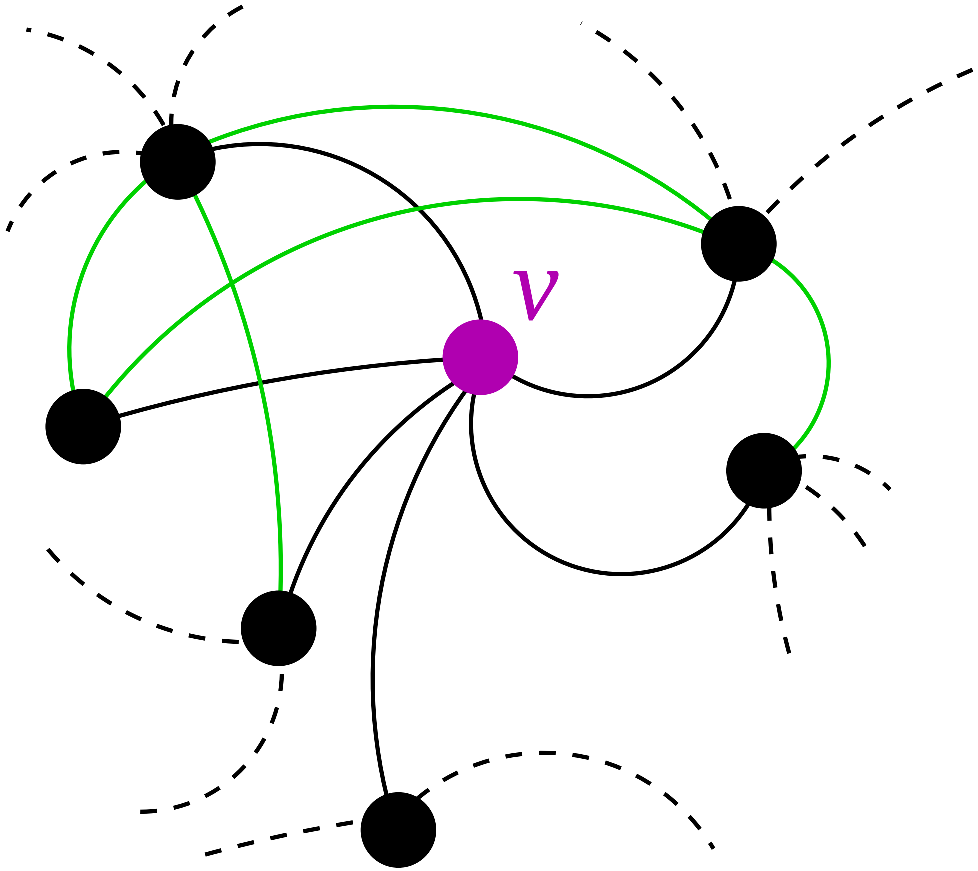
Definition 4 *The neighborhood of a vertex v, $\Gamma_v = \{i : d(i,v) = 1\}$ (so $v \notin \Gamma_v$).*

Definition 5 *The local cluster coefficient, C_v, is:*

$$C_v = |E(\Gamma_v)| / \binom{k_v}{2}$$

where $|E(\cdot)|$ gives a subgraph's total number of edges.

Definition 6 *The cluster coefficient, C, is C_v averaged over all vertices.*



Neighborhood of v with $k_v = 6$ and $|E(\Gamma_v)| = 4$, giving $C_v = 4/15$.

Real World Graphs

The Kevin Bacon Graph (KBG). Vertices are actors in IMDb (*http://www.imdb.com*), an edge between v and v' means that both v and v' has acted in a specific movie.

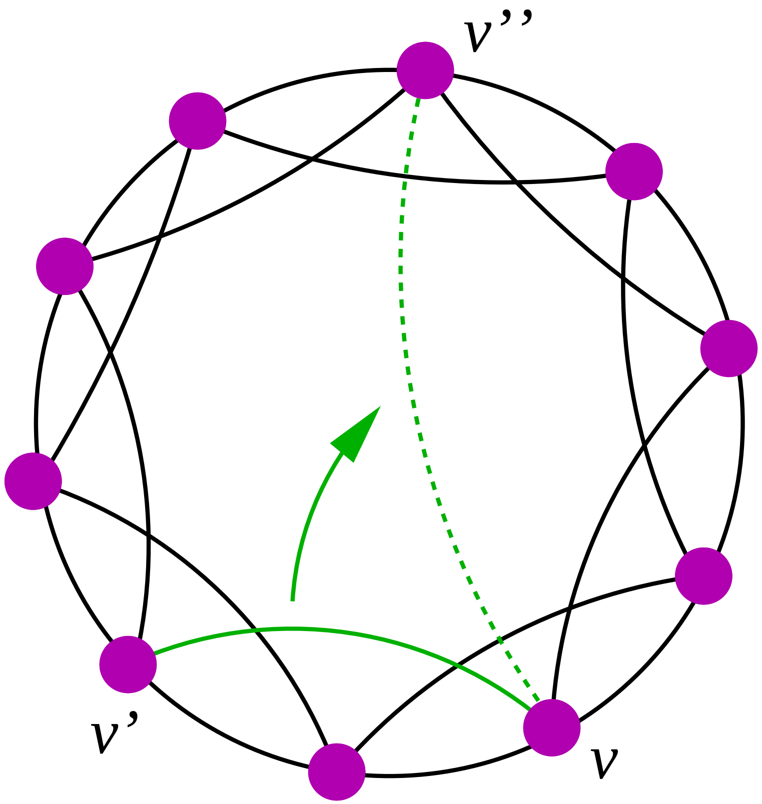
The Western States Power Grid (WSPG). Edges are high-voltage power lines west of the Rocky Mountains. Vertices are transformers, generators, substations etc.

The C. Elegans Graph (CEG) The neural network of the worm *Caenorhabditis Elegans*, with nerves as edges and synapses as vertices.

	KBG	**WSPG**	**CEG**
n	225,226	4,941	282
k	61	2.67	14
L	3.65	18.7	2.65
C	0.79 ± 0.02	0.08	0.28

Furthermore, as Beom Jun showed last week, the large k-tail of the connectivity distribution shows algebraic scaling.

Watts and Strogatz Model



1-lattice with $k = 2$ being rewired.

- Start with a 1-lattice with k-edges per vertex.
- Iterate the following for the $nk/2$ edges:

  1. Detach the v'-end of the edge from v to v' with probability p.
  2. Rewire to any other vertex v'' that is not already directly connected to v with equal probability.

If $n \gg k \gg \ln n \gg 1$ then:

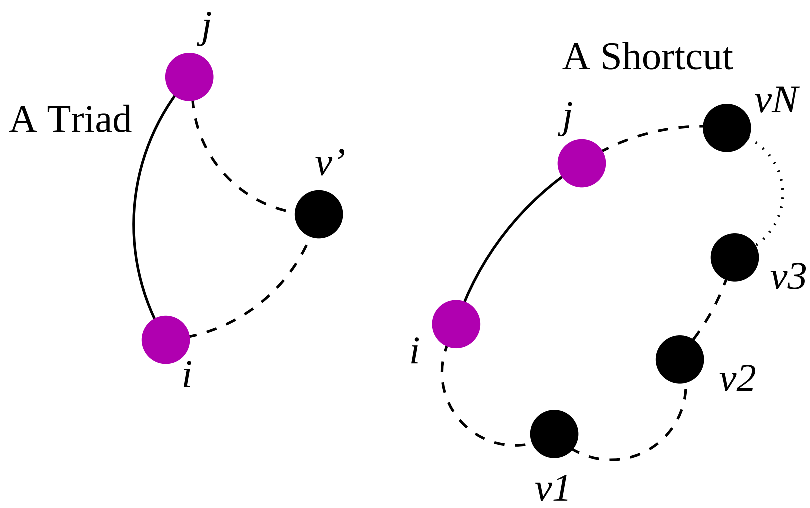
- $L \sim n/2k$ and $C \sim 3/4$ for $p \approx 0$.
- $L \sim \ln n/\ln k$ and $C \sim k/n$ for $p \approx 1$.
- $L \sim \ln n/\ln k$ and $C \sim 3/4$ for $0.001 < p < 0.01$.

The last point shows the *small world property* logarithmic $L(n)$ and high clustering.

Mechanisms for Small World Formation

The mechanisms for small world formation (in the generation algorithm) is the adding of *shortcuts* and *contractions*.

Definition 7 *The range of an edge $R(i, j)$ is the length of the shortest path between i and j in the absence of that edge.*



Definition 8 *An edge (i, j) with $R(i, j) > 2$ is called a shortcut. If $R(i, j) = 2$, (i, j) is a member of a triad.*

A model independent parameter:

Definition 9 *Given a graph of $M = kn/2$ edges, the fraction of those edges that are shortcuts is denoted by ϕ.*

Rewiring with the constraint that ϕ is fixed defines ϕ-graphs.

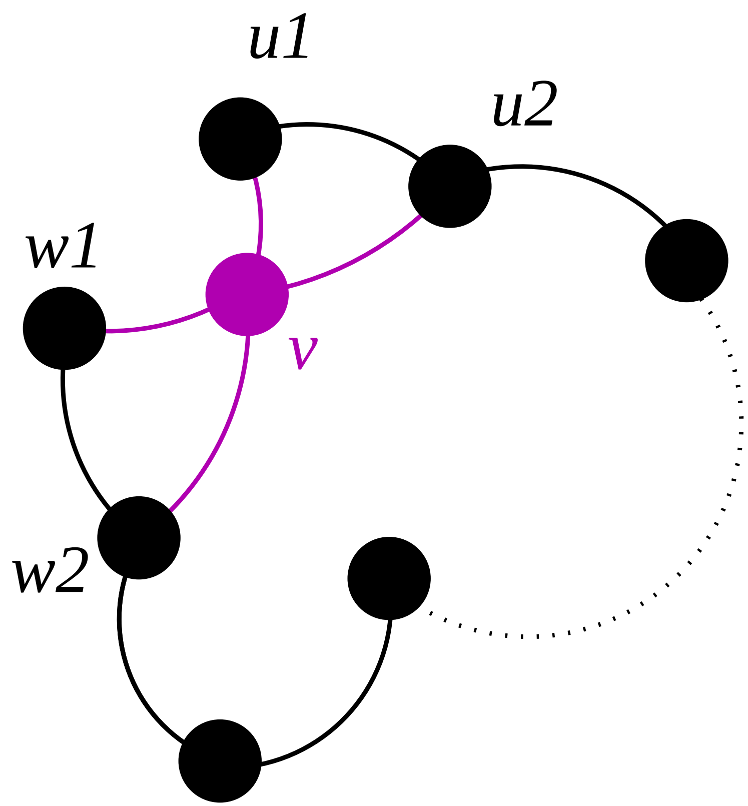
Conjecture 1 *ϕ-graphs with constant $\phi = \phi_0 > 0$, $n > 2/k\phi_0$ and $n \gg k \gg 1$ will have logarithmic length scaling.*

Slightly more general than the shortcuts:

Definition 10 *If two vertices* u *and* w *are both elements of the same neighborhood* $\Gamma(v)$*, and the shortest path length not involving edges adjacent with* v *is denoted* $d_v(u, w) > 2$*, then* v *is said to contract* u *and* w*, and the pair* (u, w) *is said to be a contraction.*

Definition 11 ψ *is the fraction of all pairs of vertices that are not connected and have one and only one common neighbor.*

ψ is for contractions what ϕ is for shortcuts. There is no known way of constructing ψ-graphs.



A contractor v, in a situation without shortcuts.

Small World Behavior Emerging from Optimization

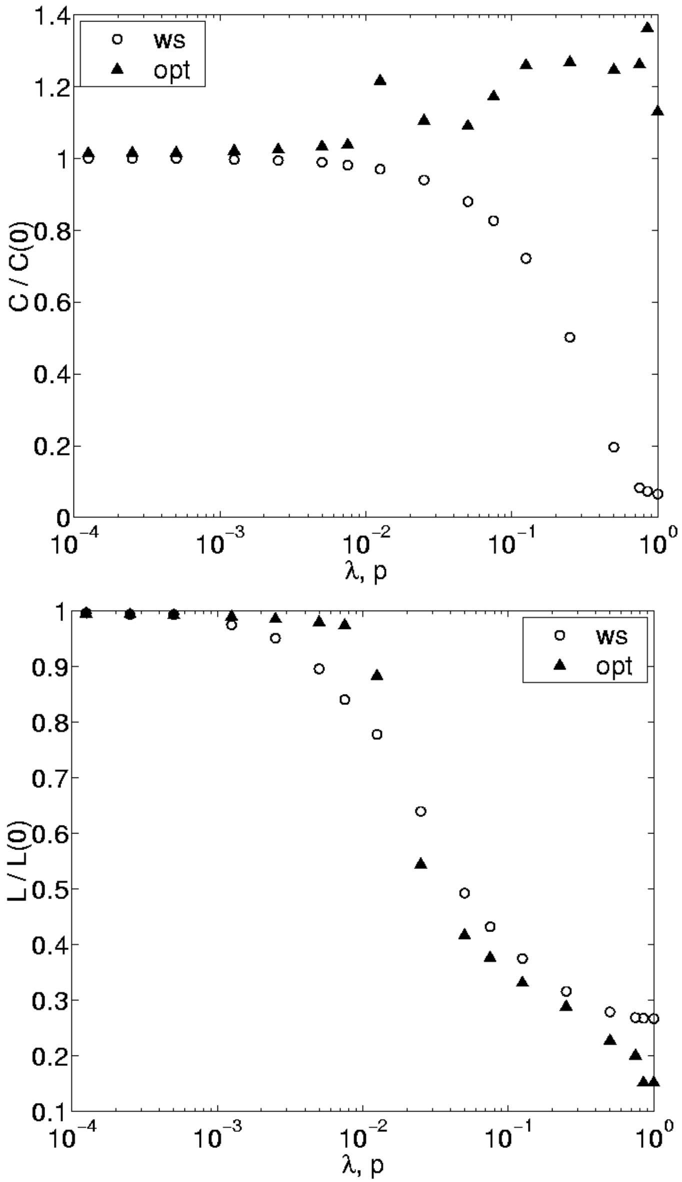
N. Mathias and V. Gopal, *Small Worlds: How and Why?*, Phys. Rev. E **63**, 21117 (2001).

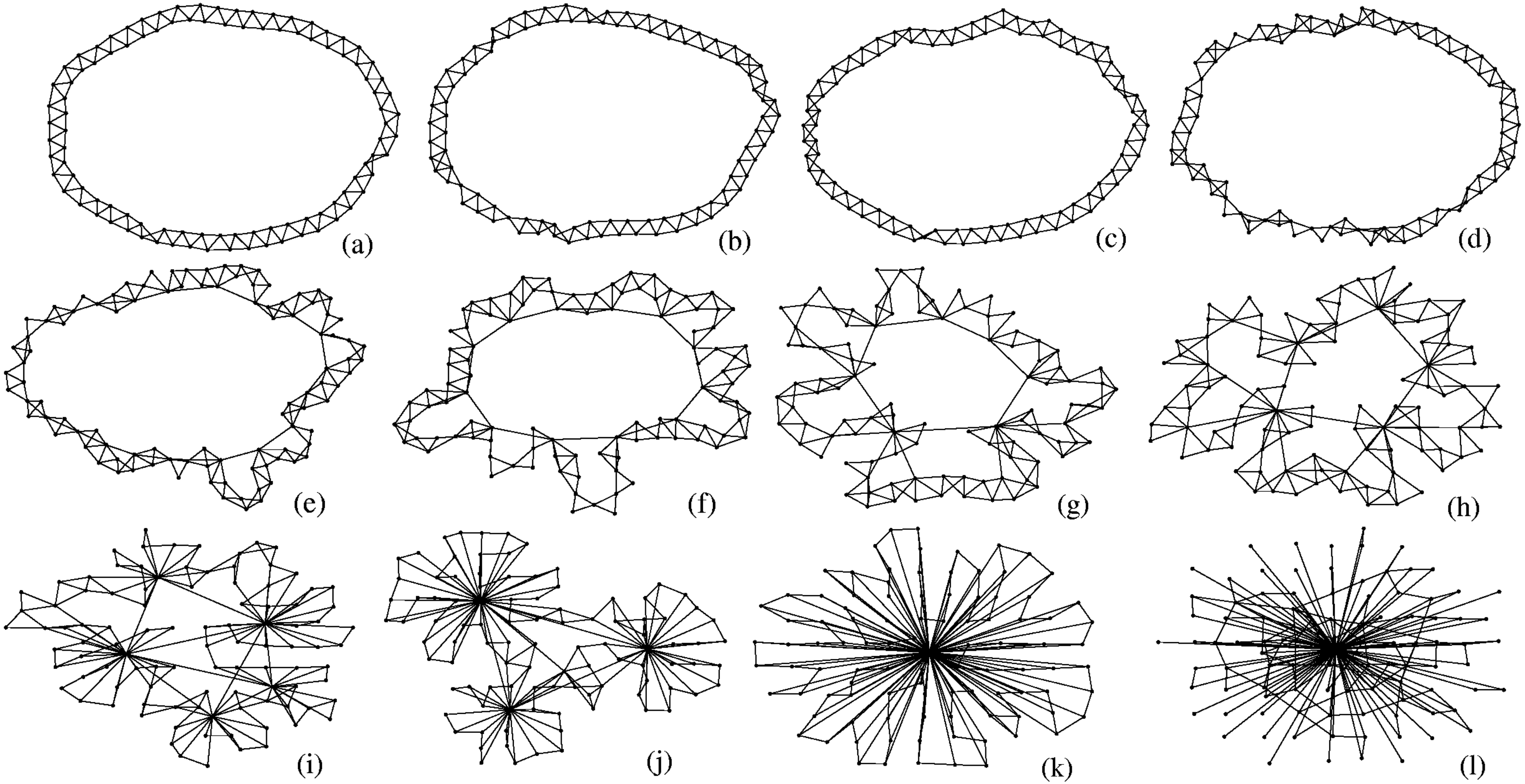
If we introduce a cost function $E = \lambda\, L + (1 - \lambda)\, W$ with

$$W = \sum_{(i,j)} \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

does low energy states correspond to small world networks? For what values of λ does this happen?

- L drops for $\lambda \approx 10^{-2}$.
- C remains $\sim$constant for all λ.
- Hubs appear and merge as λ grows.





(a) $\lambda = 0$. (b) $\lambda = 5 \times 10^{-4}$. (c) $\lambda = 5 \times 10^{-3}$. (d) $\lambda = 0.0125$. (e) $\lambda = 0.025$. (f) $\lambda = 0.05$. (g) $\lambda = 0.125$. (h) $\lambda = 0.25$. (i) $\lambda = 0.5$. (j) $\lambda = 0.75$. (k) $\lambda = 1$.

Ising Model on a 1D lattice with random long-range bonds

M. Gitterman, *Small-World Phenomena in Physics: The Ising Model*, J. Phys. A **33**, 8373.

Considers a 1-lattice with $k = 2$ (a one-dimensional cubic lattice with PBC), with additional long-range edges added with probability p.

For $p = 0$ this model have $C = 0$, for any p $C < C_{\mathrm{random}}$ (my guess), so this model might have logarithmic length scaling but *not* high clustering. (And is thus not a small world graph.)

Through transfer matrix calculations the following is found:

- With $p \in \mathcal{O}(1/n)$ long range edges the system have a finite T transition.
- If the long range edges represents annealed disorder, a finite T phase transition occurs if $p < p_{\mathrm{min}} < 1$.
- If the long range edges represents quenched disorder, a finite T phase transition occurs if $p < p_{\mathrm{min}} \approx 1$.