Solutions to Space Physics Exam, 2003–01–03.

1. When considering a satellite in transfer orbit it is reasonable to neglect air resitance and gravitation. From the rocket equation, with Δv antiparallel to v_e, we obtain

$$\Delta v = -v_e \int_0^{t_b} \frac{\dot{m}}{m}\, dt = v_e \ln\left(\frac{m_p + m_f}{m_p}\right)$$

where m_f is the mass of the required fuel. Solving for m_f we find

$$m_f = m_p \left(\exp\left(\frac{\Delta v}{v_e}\right) - 1\right)$$

Inserting $m_p = 500$ kg, $\Delta v = 3000$ m/s,and $v_e = 2000$ m/s we find $m_f = 500(\exp(1.5) - 1)$ kg $= 1741$ kg.

Answer: The satellite must carry 1741 kg of fuel.

2. Energy transport in the Sun is described in the Lecture notes on Space Physics, section 4.2.1, page 32.

3. The magnetic moment of an electron with velocity$v$ and pitch angle α is $\mu = \frac{mv^2 \sin^2 \alpha}{2B}$. Since the magnetic moment and the kinetic energy are both conserved when the electron moves in a slowly varying magnetic field it follows that

$$\sin^2 \alpha_E = \frac{B_E}{B_M}$$

where α_E is the equatorial pitch angle, B_E is the equatorial magnetic field, and B_M is the magnetic field at the mirror point where $\alpha = 90°$. The latitude of the mirror point is obtained from $r(\lambda_M) = LR_E \cos^2 \lambda_M = r_M$, so that $\cos^2 \lambda_M = r_M / r_E$. From the given formula we obtain the ratio between the equatorial magnetic field $B_E = B(L = 7, \lambda = 0)$ and the magnetic field at the mirror point $B_M = B(L = 7, \lambda_M)$ as

$$\frac{B_E}{B_M} = \frac{\cos^6 \lambda_M}{(1 + 3\sin^2 \lambda_M)^{1/2}}$$

Using $\cos^2 \lambda + \sin^2 \lambda = 1$ we find

$$\sin^2 \alpha_E = \frac{\cos^6 \lambda_M}{(4 - 3\cos^2 \lambda_M)^{1/2}} = \frac{(r_M/r_E)^3}{(4 - 3r_M/r_E)^{1/2}}$$

With $r_M/r_E = 1.4/7 = 0.2$ we numerically find $\sin^2 \alpha_E = 0.2^3/\sqrt{3.4}$ and $\alpha_E = 3.77°$.

Answer: The equatorial pitch angle is 3.8°.

4. The ozone layer is treated in the Lecture Notes on Space Physics, section 8.2.3, pages 81–83.

5. Notice that $\mathbf{B}$, $\mathbf{E}$, and $\mathbf{U}$ are all perpendicular to each other. Assume that the width of the flux tube is δ_E in the direction of $\mathbf{E}$ and δ_U in the direction of $\mathbf{U}$. Since there are no field-aligned electric fields the potential difference $\Delta\phi = E\,\delta_E$ will be constant along the flux tube, although $\mathbf{E}$ and δ_E both depend on B. The magnetic flux $\Phi = B\,\delta_U\delta_E$ in the flux tube is also constant. From the problem text we have that $U = E/B$. The plasma will drift the distance δ_U across the flux tube in the time

$$T = \frac{\delta_U}{U} = \frac{B\,\delta_U}{E} = \frac{B\,\delta_U\delta_E}{E\delta_E} = \frac{\Phi}{\Delta\phi}$$

and since Φ and $\Delta\phi$ are constant along the flux tube, T must also be constant. Q. E. D.

6. Choose a coordinate system with $\hat{\mathbf{x}}$ pointing to the Earth, $\hat{\mathbf{y}}$ to the evening, and $\hat{\mathbf{z}}$ to the north. The main magnetic field component is then $B_x(\pm z_l) = \pm 10$ nT, positive above the neutral sheet (at $z = z_l$) and negative below ($z = -z_l$). Here, z_l is a point in the tail lobe, outside the neutral sheet. Within the neutral sheet there is a small magnetic field component $B_z = 2$ nT. The tail diameter is $D = 30$ $\mathrm{R_E}$ and the cross-tail potential difference $V = 70$ kV corresponds to an electric field $\mathbf{E} = \hat{\mathbf{y}}E_y = \hat{\mathbf{y}}V/D$.

a. The power dissipated per $\mathrm{R_E}$ is $P = VI$, where the current I is determined by integrating 1 $\mathrm{R_E}$ in the $\hat{\mathbf{x}}$ direction along the tail from some point x_0 in the relevant part of the tail, and in the $\hat{\mathbf{z}}$ direction from a point $-z_l$ below the neutral sheet to z_l above as

$$I = \int_{x_0}^{x_0+\mathrm{R_E}} dx \int_{-z_l}^{z_l} dz\, j_y(z) \tag{1}$$

The reversal of the magnetic field component B_x is caused by the current density j_y in the neutral sheet. From Ampere's law (with $\partial_t\mathbf{E} = 0$) we have

$$\partial_z B_x = \mu_0 j_y$$

and inserting this in (1) we find

$$I = \frac{\mathrm{R_E}}{\mu_0}\int_{-z_l}^{z_l} dz\,\partial_z B_x = \frac{2\mathrm{R_E}}{\mu_0}B_x(z_l)$$

and the dissipated power is

$$P = \frac{2\mathrm{R_E}}{\mu_0}B_x(z_l)V$$

Numerically we obtain

$$P \approx \frac{2\cdot 6.37\cdot 10^6}{4\pi\cdot 10^{-7}}\,10\cdot 10^{-9}\cdot 70\cdot 10^3\ \mathrm{W} \approx 7\cdot 10^9\ \mathrm{W}$$

b. The flow velocity $\mathbf{U}$ of the plasma is determined by

$$\mathbf{U} = \frac{\mathbf{E} \times \mathbf{B}}{B^2}$$

The inflow from the lobes is the $\hat{\mathbf{z}}$ component of $\mathbf{U}$, or

$$U_z(z) = -E_y/B_x(z) = -\frac{V}{D\,B_x(z)}$$

Since B_x has the same sign as z the flow is always towards the neutral sheet. Numerically its magnitude is

$$U_z \approx \frac{70 \cdot 10^3}{30 \cdot 6.37 \cdot 10^6\; 10 \cdot 10^{-9}}\,\mathrm{m/s} \approx 36.6\,\mathrm{km/s}$$

c. The sunward flow in the neutral sheet is the $\hat{\mathbf{x}}$ component of $\mathbf{U}$, determined by the neutral sheet magnetic field B_z. We find

$$U_x = \frac{E_y}{B_z} = \frac{V}{D\,B_z}$$

and inserting the given numbers

$$U_x \approx \frac{70 \cdot 10^3}{30 \cdot 6.37 \cdot 10^6\;\; 2 \cdot 10^{-9}}\,\mathrm{m/s} \approx 183\,\mathrm{km/s}$$