Lösningar till tentamen i rymdfysik, 2000–10–27

1. The motion of the rocket is described by

$$\dot{\mathbf{v}}(t) = \frac{\dot{m}}{m(t)}\mathbf{v}_e + \mathbf{g}$$

During the time interval $0 < t < t_b$ the rocket mass $m(t)$ changes from m_l to $m_r = m_l - m_b$, and during this time $\dot{m} < 0$. At later times, $t > t_b$, we have $m(t) = m_r$ and $\dot{m} = 0$. Integrating the equation of motion up to time T with $\mathbf{v}(0) = 0$ gives

$$\begin{aligned}\mathbf{v}(T) &= \int_0^T \mathbf{v}_e \frac{\dot{m}}{m(t)} + \mathbf{g}\,dt \\ &= \mathbf{v}_e \int_0^{t_b} d_t(\ln m(t))\,dt + \mathbf{g}T = \mathbf{v}_e \ln\left(\frac{m_r}{m_l}\right) + \mathbf{g}T\end{aligned}$$

We choose the velocity positive upwards and insert the given values $v_e = -2000$ m/s, $T = 30$ s, $m_l = 400$ kg, and $m_b = 200$ kg. Using $g = -9.81$ kg m/s^2 and $m_r = 200$ kg we find

$$v(T) = [-2000 \ln \frac{1}{2} - 9.81 \cdot 30]\ \mathrm{m/s} = 1092\ \mathrm{m/s}$$

Answer: After 30 s the velocity of the rocket is 1092 m/s.

2. The Chapman layer is derived in section 8.3.2, page 84, in the lecture notes. A qualitative description should include the importance of the balance between the intensity of the downgoing radiation and the decreasing density of the neutral plasma with altitude for the ionisation rate. Loss of free electrons by recombination should also be discussed.

3. Assuming spherical symmetry, the mass flux due to the solar wind is

$$\phi = 4\pi R^2 n m_p v_{SW}$$

where $R = 1AU = 1.496 \cdot 10^{11}$ m, the solar wind density $n = 7 \cdot 10^6$ protons/m^3, the proton mass $m_p = 1.67 \cdot 10^{-27}$ kg, and the solar wind velocity is $v_{SW} = 450$ km/s. The mass lost during a time t is then $m_{SW} = \phi t$.

The power lost by electromagnetic radiation is

$$P = 4\pi R^2 \varepsilon$$

where $\varepsilon = 1.39$ kW/m^2 is the solar constant. The energy lost during time t is $E = Pt$. Using $E = mc^2$, this corresponds to a mass loss $m_{rad} = 4\pi R^2 \varepsilon / c^2\, t$.

The entire solar mass $M_\odot = 2 \cdot 10^{30}$ kg has been consumed when

$$m_{SW} + m_{rad} = M_\odot = 4\pi R^2 \left(\frac{\varepsilon}{c^2} + n m_p v_{SW} \right) t$$

From this we find the time

$$t = \frac{M_\odot}{4\pi R^2 \left(\frac{\varepsilon}{c^2} + n m_p v_{SW} \right)}$$

Numerically, we find that $\varepsilon / c^2 \approx 1.54 \cdot 10^{-14}$ kg m^{-2} s^{-1} is comparable to $n m_p v_{SW} \approx 5.26 \cdot 10^{-15}$ kg m^{-2} s^{-1}. The Sun will have lost all its mass after $3.4 \cdot 10^{20}$ s, or about 10^{13} years.

4. When an electron moves in a slowly varying magnetic field, its kinetic energy and magnetic moment are conserved. Hence, the kinetic energy will be $E_k = 10$ eV $= 1.6 \times 10^{-18}$ J also in the equatorial plane. The velocity is then $v = \sqrt{2E_k/m_e}$, where $m_e = 9.1 \cdot 10^{-31}$ kg is the electron mass. Introducing the velocity components parallel and perpendicular to the magnetic field, $v_\parallel = v\cos\alpha$ and $v_\perp = v\sin\alpha$, we can express the magnetic moment as

$$\mu = \frac{m_e v_\perp^2}{2B} = \frac{m_e v^2}{2B}\sin^2\alpha = E_k\frac{\sin^2\alpha}{B}$$

Since μ and E_k are constants, $\sin^2\alpha/B$ must be a constant. Using that $\sin^2\alpha = 1$ when $B = B_I$ we find $v_\perp^2 = v^2\sin^2\alpha = v^2 B/B_I$. The parallel velocity at a point where the magnetic field is $B \leq B_I$ may then be written

$$v_\parallel = \sqrt{v^2 - v_\perp^2} = \sqrt{v^2(1 - B/B_I)}$$

In the equatorial plane where $B = B_E = 5\,\mu$T the parallel velocity $v_{\parallel E}$ is

$$v_{\parallel E} = \sqrt{\frac{2E_k}{m_e}(1 - B_E/B_I)}$$

Numerically,

$$v_{\parallel E} = \sqrt{\frac{2 \cdot 1.6 \cdot 10^{-18}}{9.1 \cdot 10^{-31}}(1 - 5/50)}\ \text{m/s} = 1.78 \cdot 10^6\ \text{m/s}$$

Answer: At the equatorial plane the velocity of the electron is $1.78 \cdot 10^6$ m/s

5. One derivation of the plasma frequency is given in section 5.2.1, page 41, in the lecture notes, but there are others. The (angular) plasma frequency is given by

$$\omega_{pe} = \sqrt{\frac{ne^2}{\varepsilon_0 m_e}}$$

If the electron density is $n = 10^{12}$ m^{-3}, we find

$$\omega_{pe} = 1.6 \cdot 10^{-19}\sqrt{\frac{10^{12}}{8.85 \cdot 10^{-12}\, 9.1 \cdot 10^{-31}}}\ \text{s}^{-1} = 5.64 \cdot 10^7\ \text{s}^{-1}$$

or $f_p = \omega_{pe}/2\pi \approx 9$ MHz. Since the O^+ ions are about $16 \cdot 1837$ times heavier than electrons, they can be neglected. If we include them we find

$$\omega_p = \sqrt{\frac{ne^2}{\varepsilon_0}\left(\frac{1}{m_e} + \frac{1}{m_{O^+}}\right)} \approx \omega_{pe}(1 + 1.7 \cdot 10^{-5})$$

which is very close to ω_{pe}.

6. Using the coordinate system indicated in the Figure, we find $F_y = j_x B_G$, where $B_G = 50$ nT is the magnetic field strength in the equatorial generator region. This gives the perpendicular current density

$$j_x = \frac{F_y}{B_G}$$

The parallel current can be calculated from $\partial_{\mathbf{r}} \cdot \mathbf{j} = \partial_x j_x + \partial_z j_z = 0$. Assuming symmetry about the equatorial plane we have $j_z = 0$ at $z = 0$, and at a point z_G above the generator region we have

$$j_z(x, z_G) = -\int_0^{\infty} \partial_x j_x \, dz = -\int_0^{\infty} \frac{\partial_x F_y}{B_G} \, dz$$

Using that $\int_0^{\infty} \exp -z^2/L_z^2 \, dz = L_z\sqrt{\pi}/2$ and calculating the derivative of F_y as $\partial_x \exp(-x^2/L_x^2) = -2x/L_x^2 \exp(-x^2/L_x^2)$ we find

$$j_z(x, z_G) = \sqrt{\pi}\frac{L_z x F_0}{L_x^2 B_G} \exp\left(-\frac{x^2}{L_x^2}\right)$$

a. To find the maximum parallel current density we put $\partial_x j_z = 0$, which yields $1 - 2x^2/L_x^2 = 0$, or $x^2 = L_x^2/2$. Inserting this we find

$$|j_z(z_G)|_{max} = \sqrt{\frac{\pi}{2}}\frac{L_z F_0}{L_x B_G} \exp\left(-\frac{1}{2}\right)$$

Putting in the numbers we find

$$|j_z(z_G)|_{max} = \sqrt{\frac{\pi}{2e}} \cdot \frac{500 \cdot 10^3 \, 10^{-17}}{50 \cdot 10^3 \, 50 \cdot 10^{-9}} \text{ A/m}^2 \approx 1.52 \text{ nA/m}^2$$

Answer: The maximum current density is 1.52 nAm^{-2}.

b. Outside the generator region the current is strictly field-aligned, which from $\partial_{\mathbf{r}} \cdot \mathbf{j} = 0$ means that the total current in a flux tube is constant. The area of a flux tube is inversely proportional to B, so this implies that j_z/B is constant along the field line. The ionospheric current density is then $j_z(z_I) = j_z(z_G)B_I/B_G$, where $B_I = 50\ \mu$T and z_I indicates a point at 2000 km altitude. If the electron density is n, the current density is related to the velocity v_z by $j_z = env_z$, and the kinetic energy K can be expressed as

$$K = \frac{mv_z^2}{2} = \frac{m}{2}\left(\frac{j_z(z_I)}{en}\right)^2 = \frac{m}{2}\left(\frac{j_z(z_G)B_I}{enB_G}\right)^2$$

Numerically we find

$$K \approx \frac{9.1 \cdot 10^{-31}}{2}\left(\frac{1.52 \cdot 10^{-9} \, 5 \cdot 10^{-7}}{1.6 \cdot 10^{-19} \, 10^5 \, 5 \cdot 10^{-10}}\right)^2 \text{ J} \approx 4.1 \cdot 10^{-15} \text{ J} \approx 25.6 \text{ keV}$$

Answer: The kinetic energy of the current carrying electrons is 25.6 keV.