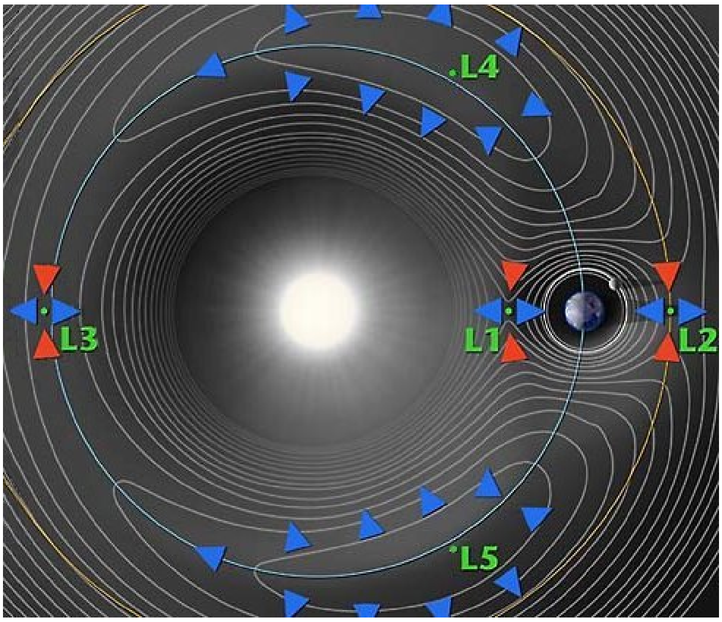
Umeå Universitet
Institutionen för fysik

The Three-Body Problem and the Lagrangian Points

-A report in the course Space Physics-



http://en.wikipedia.org/wiki/Image:Lagrange_points.jpg, also cf. [2]

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I Introduction

Space Physics may be defined as the region of the universe where astrophysical phenomena can be studied *in situ*, i.e., by direct measurements from spacecrafts or satellites.[1] From this it follows that in these days the named region is limited to the Solar System.

The Solar System, being the home for mankind in the universe, inhabits besides Earth seven more planets, a total of more than one hundred moons orbiting these planets as well as some dwarf planets and an uncountable amount of asteroids. As being neutrally charged, all these bodies only interact with one another due to gravitational forces. Accordingly, from the point of Celestial Mechanics, determination of the trajectory of a given body would be impossible if one could not neglect the influences of almost all these bodies on the given one as a result of the nature of the gravitational force being proportional to the mass of and the inverse squared distance to these bodies. To come full circle, such a given body can of course be a spacecraft or a satellite so that Celestial Mechanics is also an important topic in Space Physics.

But nevertheless, the only analytically solvable problems in Celestial Mechanics are the two-body (Kepler's problem) and partly the circular restricted three-body problem. The last one shall now be the main topic in this report as nature offers us with it a great opportunity for satellite measurements by making use of the Lagrangian points.

II The Circular Restricted Three-Body Problem

II.1 Preliminary Remarks

The three-body problem refers to three bodies which move under their mutual gravitational attraction. There does not exist a general analytical solution to this problem but chaotic solutions and numerical ones based on iterative methods.

If two of the three bodies move in circular and coplanar orbits around their common barycentre, and additionally, the third mass is small compared to the other two masses so that the third body does not affect the movements of the other bodies one speaks of the circular restricted three-body problem. In this case the two big masses move on orbits which are determined by the solution to the two-body problem and the remaining assignment is to make predictions for the

1 cf. Rönnmark, Kjell: Lecture Notes on Space Physics, p. 1.

trajectory of the third body being influenced by the gravitational field of the two big masses. This regime describes a good approximation for certain systems in our Solar System like a spacecraft in the gravitational field of Earth and the Sun or a space probe flying to Jupiter or Saturn being exposed to the gravitational attraction of the Sun and the planet. An analytical solution to the behaviour of the third mass can be given in the case of equilibrium points (the Lagrangian points) and a description of the movement can be achieved as for example in the problem of the Hill sphere or the zero-velocity surfaces.

II.2 The Two-Body Problem

As some terms of the solution to the two-body problem are used in the discussion of the circular restricted three-body problem a short overview of this solution shall be given here.

With the definition

$$\mu := G(m_1+m_2) \quad , \tag{1}$$

where *G* is the gravitation constant, m_1 and m_2 are the masses of the two bodies, respectively, Newton's second law for the two-body problem reads

$$\ddot{\vec{r}} = -\mu \frac{\vec{r}}{r^3} \quad . \tag{2}$$

Note, that eq.(1) is the equation of motion for the vector $\vec{r}$ which denotes the relative position between the two bodies. This equation can be solved giving the modulus of the relative position depending on the polar angle φ,

$$r(\phi) = \frac{a(1-e^2)}{1+e\cos\phi} \quad , \tag{3}$$

in which *a* is the semi-major axis and *e* the eccentricity of the elliptical orbit. By some manipulations of this equation and integration over one period[2] of the orbit you can determine the frequency *n* of the orbit to

$$n = \sqrt{\frac{\mu}{a^3}} \quad . \tag{4}$$

This frequency is also referred to as the mean motion.

2 cf. Murray, Dermott: Solar System Dynamics, pp.29-30 and [9].

II.3 Equations Of Motion

Let us consider a reference frame with its origin in the common centre of mass and the x-axis along the reference line of the two heavy bodies which rotates uniformly with the angular velocity *n* (the mean motion of the heavy bodies) around the *z*-axis. In this reference frame the positions of the two heavy masses remain fixed. With the assumption that $m_1 > m_2$ it is now convenient to rescale the unit of mass so that $\mu = G(m_1+m_2) = 1$ and to define

$$\bar{\mu} := \frac{m_2}{m_1+m_2} \quad . \tag{5}$$

Furthermore, we introduce $\tilde{\mu}_1 := G m_1$ and $\tilde{\mu}_2 := G m_2$, or in the rescaled units $\mu_1 = \frac{\tilde{\mu}_1}{\mu} = 1-\bar{\mu}$ and $\mu_2 = \bar{\mu}$, respectively. With a further rescale of the unit of length so that $a = 1$ we see that $\mu_1+\mu_2 = a = 1$. Now we are able to adjust the coordinate system in such a way that the body m_1 has the coordinates (*x*,*y*,*z*) = ($-\mu_2$,0,0) and the body m_2 the coordinates (*x*,*y*,*z*) = (μ_1,0,0).

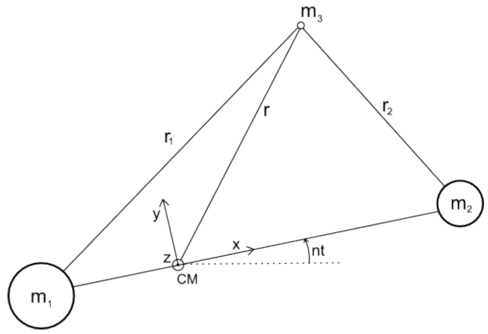

Fig.1. A planar view of the introduced reference frame for the circular restricted three-body problem

The equations of motions in this reference frame for the light body at the position $\vec{r} = (x, y, z)$ then read

$$\ddot{x} - 2n\dot{y} - n^2 x = -\left(\mu_1 \frac{x+\mu_2}{r_1^3} + \mu_2 \frac{x-\mu_1}{r_2^3}\right) \quad , \tag{6a}$$

$$\ddot{y} + 2n\dot{x} - n^2 y = -\left(\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3}\right) y \quad , \tag{6b}$$

$$\ddot{z} = -\left(\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3}\right) z \quad . \tag{6c}$$

In this set of equations we see that there is a force due to the gravitational field of the heavy masses on the right hand side. But there are also terms on the left hand side which indicate the centrifugal force being proportional to the squared mean motion and the force of the Coriolis effect which is proportional to the body's velocity and the mean motion. As a result of the Coriolis effect the movement of the light body in the plane of the heavy masses' orbits is coupled whereas the movement in the *z*-direction decouples from the *x*- and *y*-direction.

II.4 The Lagrangian Points

In order to be in an equilibrium position we have to demand that the conditions $\ddot{x} = \ddot{y} = \ddot{z} = \dot{x} = \dot{y} = \dot{z} = 0$ are simultaneously fulfilled, i.e., that all forces acting on the small body in the rotating frame of reference vanish. From this it is clear that there cannot be a Coriolis force in an equilibrium position. So we have to look for points where the gravitational force and the centrifugal force cancel each other. This can be done analytically, but as this would go beyond the scope of this report we may be content with a qualitative analysis.

By looking on the set of equations (6) we see that the condition $\ddot{z} = 0$ can only be fulfilled for $z = 0$ so that the task of finding the equilibrium positions reduces to a planar problem. Furthermore, it follows that the centrifugal force acting on the light body always points radially outwards with respect to the common barycentre of the heavy masses. This makes us come to the conclusion that there are only a few possible positions where the radially outwards pointing centrifugal force $\vec{F}_Z$ can be cancelled by the gravitational forces $\vec{F}_1$ and $\vec{F}_2$. Obviously, three of them can be found on the reference line connecting the two masses m_1 and m_2 like it is depicted in fig.2.

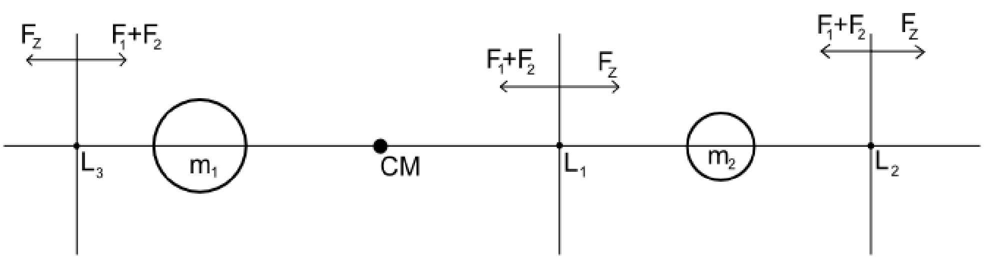

Fig.2. Positions of the Lagrangian points L_1, L_2 and L_3

On the one hand here the gravitational forces between the two major masses are directed in opposite directions and the total gravitational force at the first Lagrangian point L_1 is reduced in that way that it is oppositely equal to the centrifugal force. On the other hand there are the Lagrangian points L_2 and L_3 outside each major mass where the sum of the gravitational forces cancels the centrifugal force. But now the symmetry of the problem suggests that there have to be two more equilibrium points, each of them building an equilateral triangle with the major bodies (see fig.3). Their existence might not be seen offhand but they are also found analytically.

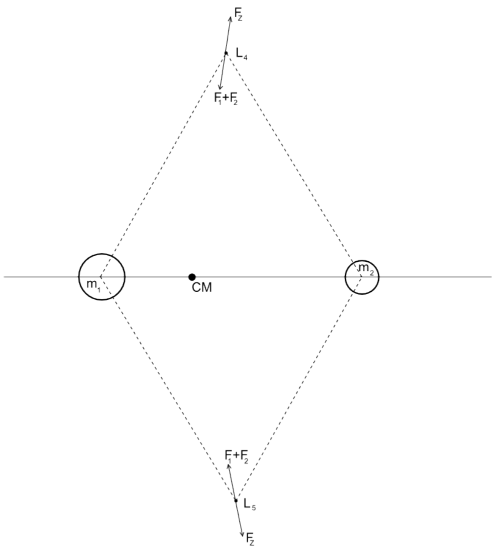

Fig.3. Positions of the Lagrangian points L_4 and L_5

As already mentioned, these qualitatively found Lagrangian points can also be derived analytically. Doing this one finds for the coordinates of the triangular points L_4 and L_5

$$x(L_{4/5}) = 1/2 - \mu_2\ ;\quad y(L_{4/5}) = \pm\frac{\sqrt{3}}{2}\quad . \tag{7}$$

The x-coordinates of the remaining equilibrium points can be given in series,

$$x(L_1) = \mu_1 - \alpha + \frac{1}{3}\alpha^2 + \frac{1}{9}\alpha^3 + \frac{23}{81}\alpha^4 + O(\alpha^5)\quad , \tag{8a}$$

$$x(L_2) = \mu_1 + \alpha + \frac{1}{3}\alpha^2 - \frac{1}{9}\alpha^3 - \frac{31}{81}\alpha^4 + O(\alpha^5)\quad , \tag{8b}$$

$$x(L_3) = \mu_1 - \left(2 - \frac{7}{12}\beta + \frac{7}{12}\beta^2 - \frac{13223}{20736}\beta^3 + O(\beta^4)\right)\quad , \tag{8c}$$

where the parameters introduced are $\alpha := \left(\frac{\mu_2}{3\,\mu_1}\right)^{1/3}$ and $\beta := \frac{\mu_2}{\mu_1}$, respectively; the *y*-coordinates are all equal to zero. Note, that all coordinates are still given in the rescaled length units – but to obtain the true values you simply need to multiply expressions (7) and (8) with the "semi-major axis" *a*.

An important topic which is always related to equilibrium positions is the question regarding the stability of these points. When asking for the linear stability of the Lagrangian points, i.e., putting the body in one of these points and considering a small perturbation, you find that the Lagrangian points L_1, L_2 and L_3 are not stable equilibrium positions. This means that no natural body will stay in these points. But the triangular Lagrangian points are stable, provided, that the mass ratio of the two heavy masses satisfies the condition $m_1/m_2 > 24.97$. A body positioned in one of these two Lagrangian points which experiences a small perturbation will then perform an oscillatory movement around the point. This is called libration and the resulting orbit in the co-rotating reference frame is referred to as an tadpole orbit. If the amplitude of this motion increases it can result in a movement which encloses both triangular Lagrangian points. One then speaks of a horseshoe orbit.

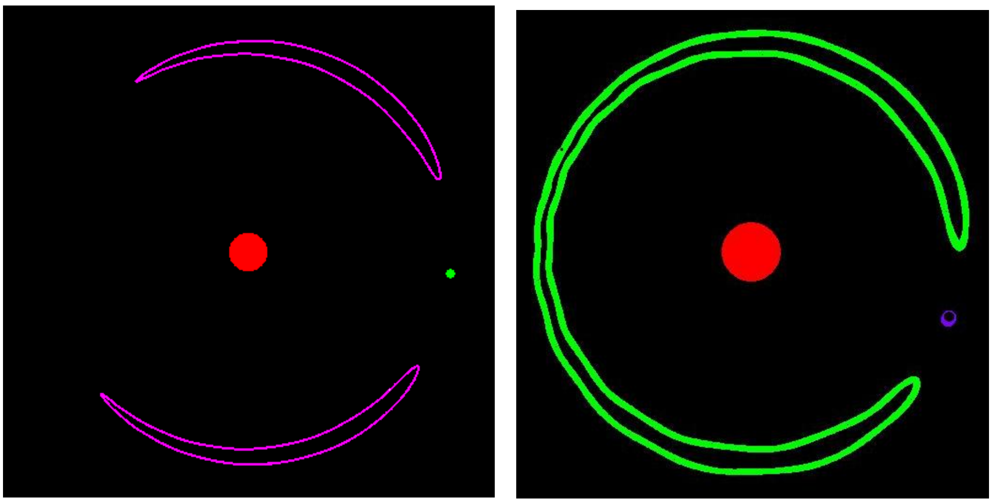

Fig.4. Tadpole orbits (on the left) and a horseshoe orbit (on the right) for a three-body system[3]

Due to the instability of the co-linear Lagrangian points no natural objects have been found at these points in any three-body system until today. But there are many examples for natural objects orbiting on or around (due to libration) the triangular Lagrangian points. The most famous of these are the Trojan asteroids of Jupiter. An example of tadpole orbits are the small moons Telesto and

3 cf. [1].

Calypso orbiting L_4 and L_5, respectively, of the system consisting of Saturn and the bigger moon Tethys. The moons Janus and Epimetheus are both on a horseshoe orbit in the Sun-Saturn system.

Finally, I like to give two examples for the use of the Lagrangian points in Space and Astrophysics.

Currently the spacecraft *SOHO* (Solar and Heliospheric Observatory) is on a halo orbit around the Lagrangian point L_1 of the Sun-Earth system which is approximately $1.5 \cdot 10^6$ km away from Earth. It is not positioned exactly at L_1 as this would not be a stable position like discussed above and for some reason it would also complicate the communication with Earth. Nevertheless, the halo orbit lies in the plane which passes through L_1 and is perpendicular to the reference line connecting Sun and Earth. Therefore, due to the weak forces in this plane near to L_1, it does not need much fuel to keep the spacecraft in its orbit. Apparently, the second advantage is that the spacecraft always stays at the same position relative to the Sun and Earth so that there are no bigger objects disturbing the view onto the Sun. The three main scientific objectives of *SOHO* are to investigate the outer layer of the Sun, to make observations of the solar wind and to probe the interior structure of the Sun. Besides that the spacecraft was able to detect many yet unknown comets which crossed the craft's field of view.

The second example is the planned James Webb Space Telescope which will be positioned at the Lagrangian point L_2 of the Sun-Earth system. The telescope's main mission is to search for light from old stars and galaxies as well as to study and understand the formation of galaxies, planetary systems and stars. Due to several effects the telescope has to operate at infrared wavelengths and accordingly has to be kept at a very low and stable temperature (about 40 K). At this point the usefulness of L_2 appears on the scene. For keeping the telescope at this low temperature a big metalized sunshield shall be used which blocks infrared radiation. But now the position at L_2 ensures that the two main sources of infrared radiation, Sun and Earth, all the time occupy roughly the same relative position in the telescope's view which will make the use of the sunshield most effective.

III Summary

We have seen that the three-body problem not only is a severe mathematical problem, it also offers with the Lagrangian points an interesting and impressive insight on Celestial Mechanics as well as it plays an important role in modern research concerning Space and Astrophysics.

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