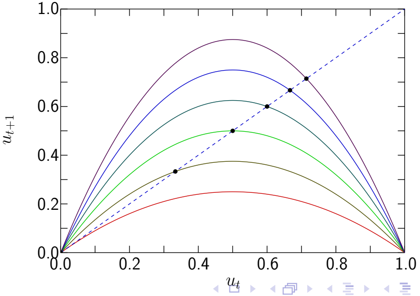
Definition

The logistic map is defined by

$$u_{t+1} = ru_t(1 - u_t), \quad 0 < r < 4.$$

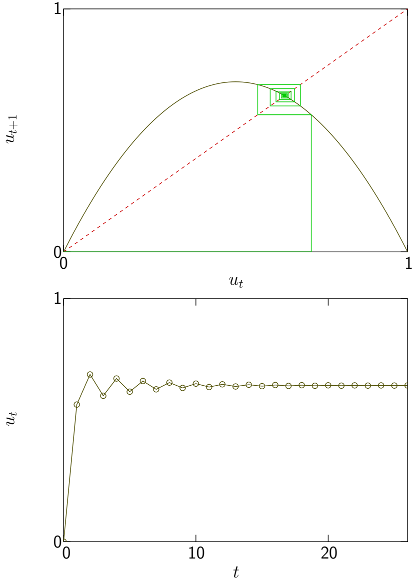
The steady states and the corresponding eigenvalues $\lambda = f'(u^*)$ are

$$u_1^* = 0, \qquad \lambda_1 = r,$$
$$u_2^* = \frac{r-1}{r}, \qquad \lambda_2 = 2 - r.$$

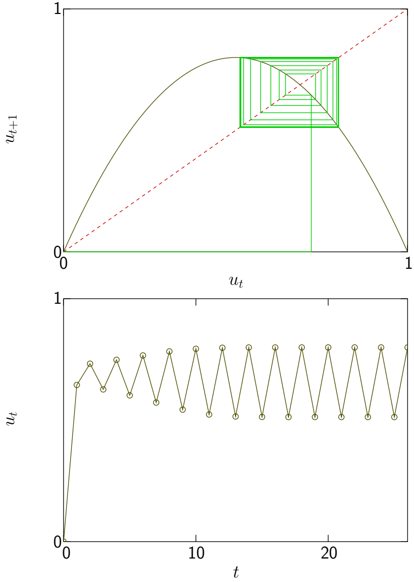


Curves for $r = 1, 1.5, \ldots 3.5$.

Stable fixed point $r = 2.8$, $\lambda = 2 - r = -0.8$



Unstable fixed point $r = 3.2$, $\lambda = 2 - r = -1.2$



To understand period doubling

Consider the map from u_t to u_{t+2} defined by

$$\begin{aligned} u_{t+1} &= r u_t(1-u_t), \\ u_{t+2} &= r u_{t+1}(1-u_{t+1}). \end{aligned}$$

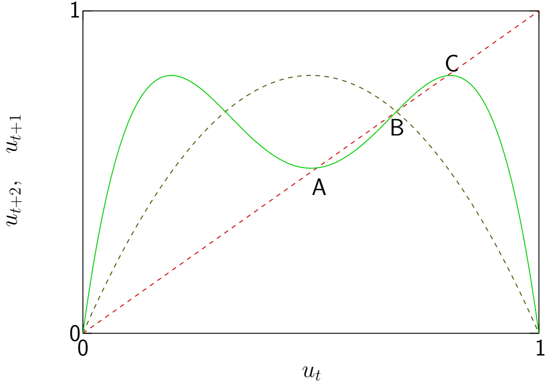
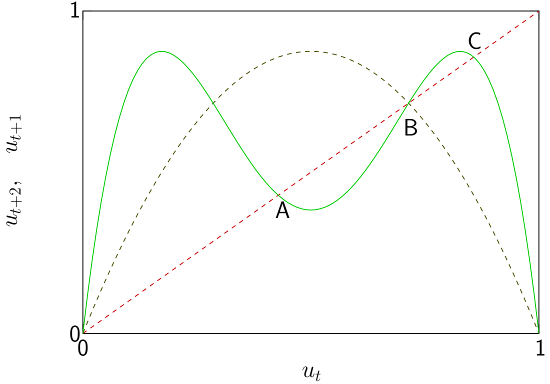


Figure for $r = 3.2$.

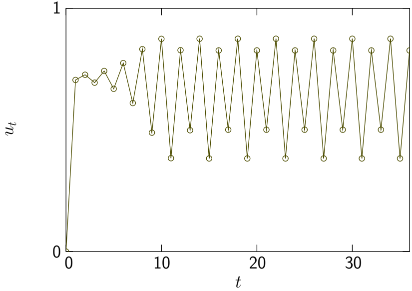
Period doubling, again!

Figure for $r = 3.5$



The behaviors at A and C are unstable, $f' < -1$!

Oscillations with period 4, $r = 3.5$.



The same reasoning may be applied for u_{t+4} which gives an oscillation of period eight. . . this period doubling may be continued without limit.

The path to chaos

- For $1 < r \leq 3$ there is a unique solution $(r-1)/r$.
- For $3 < r \leq 1+\sqrt{6}(\approx 3.45)$ the system has periodic fluctuations between two values.
- For $1+\sqrt{6} < r < 3.54$ (approximately) the system has periodic oscillations between four values.
- For $3.54 < r < 3.57$ the system oscillates between 8, 16, 32, values, etc.
- At $r \approx 3.57$ is the onset of chaos. We can no longer see any oscillations of finite period and slight variations in the initial value yields dramatically different results over time.

Period doublings and the onset of chaos