5.2.2 Complex eigenvalues

This exercise leads you through the solution of a linear system where the eigenvalues are complex. The system is

$$\begin{cases} \dot{x} &= x - y, \\ \dot{y} &= x + y. \end{cases}$$

a) Find $\mathbf{A}$ and show that it has eigenvalues $\lambda_1 = 1 + i$, $\lambda_2 = 1 - i$, with eigenvectors $\mathbf{v}_1 = (1, i)$, $\mathbf{v} + 2 = (-i, 1)$. (Note that the eigenvalues are complex conjugates, and so are the eigenvectors—this is always the case for real $\mathbf{A}$ with complex eigenvalues.

Solution: This gives

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \Rightarrow \quad \tau = 2, \quad \Delta = 2,$$

and the eigenvalues become

$$\lambda_{1,2} = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2} = 1 \pm i.$$

1. $\lambda_1 = 1 + i$: $\mathbf{v}_1 = (i, 1)$.
2. $\lambda_2 = 1 - i$: $\mathbf{v}_2 = (-i, 1)$.

b) The general solution is $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$. so in one sense we're done! But this way of writing $\mathbf{x}(t)$ involves complex coefficients and looks unfamiliar. Express $\mathbf{x}(t)$ purely in terms of real-valued functions. (Hint: Use $e^{i\omega t} = \cos\omega t + i \sin\omega t$ to rewrite $\mathbf{x}(t)$ in terms of sines and cosines, and then separate the terms that have a prefactor of i from those that don't.)

Solution:

$$\begin{aligned} \mathbf{x}(t) &= c_1 e^{(1+i)t}\begin{pmatrix} i \\ 1 \end{pmatrix} + c_2 e^{(1-i)t}\begin{pmatrix} -i \\ 1 \end{pmatrix} \\ &= e^t\left[c_1(\cos t + i\sin t)\begin{pmatrix} i \\ 1 \end{pmatrix} + c_2(\cos t - i\sin t)\begin{pmatrix} -i \\ 1 \end{pmatrix}\right] \\ &= e^t\left[(c_1 + c_2)\cos t\begin{pmatrix} 0 \\ 1 \end{pmatrix} - (c_1 + c_2)\sin t\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] \\ &+ ie^t\left[(c_1 - c_2)\cos t\begin{pmatrix} 1 \\ 0 \end{pmatrix} - (c_1 - c_2)\sin t\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] \end{aligned}$$

With $c_1, c_2 = a \pm ib$ we can allow for arbitrary initial conditions.