5.2.13 Damped harmonic oscillator

The motion of a damped harmonic oscillator is described by

$$\ddot{x} + b\dot{x} + kx = 0,$$

where $b > 0$ is the damping constant.

a) Rewrite the equation as a two-dimensional linear system.

Solution: When rewritten as two first order equations it becomes

$$\begin{cases} \dot{x} &= v, \\ \dot{v} &= -\frac{k}{m}x - \frac{b}{m}v. \end{cases}$$

b) Classify the fixed point at the origin and sketch the phase portrait. Be sure to show all the different cases that can occur, depending on the relative sizes of the parameters.

Solution: The matrix becomes

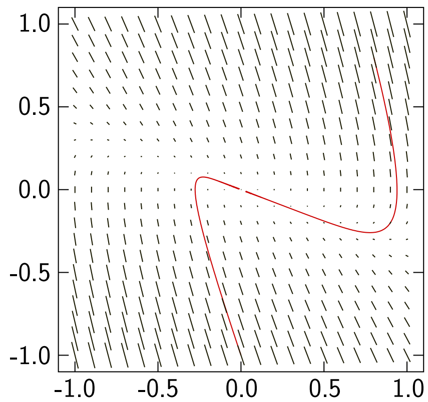
$$A = \begin{pmatrix} 0 & 1 \\ -k/m & -b/m \end{pmatrix} \quad \Rightarrow \tau = -b/m, \quad \Delta = k/m.$$

The eigenvalues become

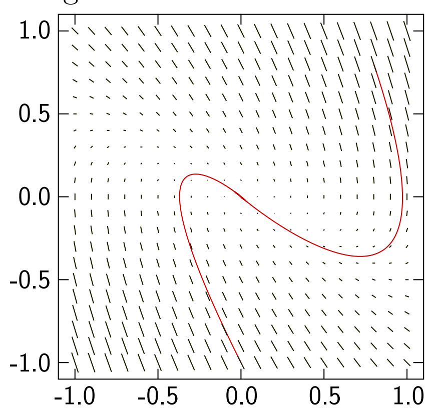
$$\lambda_{\pm} = -\frac{b}{2m} \pm \frac{1}{2}\sqrt{(b/m)^2 - 4k/m}.$$

Considering the square root there are three possibilities. We here take $m = 1$ and $k = 1$ and let b vary to get the three different cases. We illustrate the solutions both with the vector fields and with the flow starting from two initial positions.

1. Positive quantity in the square root, $(b/m)^2 > 4k/m$, which we get with $b = 3$:

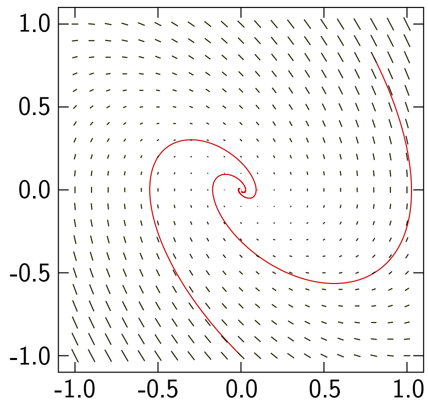


2. Vanishing quantity in the square root, $(b/m)^2 = 4k/m$, which we get with $b = 2$:



This requires some more thought and is discussed further below.

3. Negative quantity in the square root, $(b/m)^2 < 4k/m$, which we get with $b = 1$:



Note that this is a case when the eigenvectors are equal, and with only a single eigenvector.

c How do our results relate to the standard notions of overdamped, critically damped, and underdamped vibrations?

Solution: These three cases correspond to overdamped, critically damped and underdamped.

Solution for the case with just a single eigenvalue

When $b = 2$ we have $A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$ and we get $\tau = -2$ and $\Delta = 1$ which gives $\lambda_1 = \lambda_2 = \tau/2 = -1$. To determine the eigenvector, $\mathbf{v}_1$, we try

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\lambda & 1 \\ -1 & -2-\lambda \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} u+w \\ -u-w \end{pmatrix} \quad \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The second eigenvector is from $[A - \lambda\mathbf{I}]\mathbf{v}_2 = \mathbf{v}_1$ and we get

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} u+w \\ -u-w \end{pmatrix} \quad \to \quad w = 1-u.$$

We are here free to choose u as we like and can equally well get

$$\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{as} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Chosing the first we find that the full solution is

$$\mathbf{x}(t) = c_1 e^{\lambda t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\left[te^{\lambda t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{\lambda t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right].$$

With $\mathbf{x}_0 = (1,1)$ we get $c_1 = 1$ and $c_2 = 2$:

$$\mathbf{x}(t) = e^{\lambda t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2\left[te^{\lambda t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{\lambda t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right].$$

which is the same as

$$\mathbf{x}(t) = e^{\lambda t}\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2t\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]. \tag{1}$$

The more general case

To consider the general case we note that we are free to choose u as we like and we also consider a general starting point. We then write

$$\mathbf{v}_2 = \begin{pmatrix} u \\ 1-u \end{pmatrix}.$$

and we find that the full solution then becomes

$$\mathbf{x}(t) = c_1 e^{\lambda t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\left[te^{\lambda t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{\lambda t}\begin{pmatrix} u \\ 1-u \end{pmatrix}\right].$$

With $\mathbf{x}_0 = (x_0, y_0)$ we get

$$\begin{aligned} c_1 + uc_2 &= x_0 \quad \Rightarrow \quad c_1 = x_0 - uc_2, \\ -c_1 + (1-u)c_2 &= y_0 \quad \Rightarrow \quad uc_2 - x_0 + (1-u)c_2 = y_0 \quad \Rightarrow \quad c_2 = x_0 + y_0, \end{aligned}$$

and $c_1 = (1-u)x_0 - uy_0$.

$$\mathbf{x}(t) = [(1-u)x_0 - uy_0]e^{\lambda t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + (x_0+y_0)\left[te^{\lambda t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{\lambda t}\begin{pmatrix} u \\ 1-u \end{pmatrix}\right].$$

$$\begin{aligned} \mathbf{x}(t) &= e^{\lambda t}\begin{pmatrix} (1-u)x_0 - uy_0 + u(x_0+y_0) \\ (u-1)x_0 + uy_0 + (1-u)(x_0+y_0) \end{pmatrix} + (x_0+y_0)te^{\lambda t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= e^{\lambda t}[\mathbf{x}_0 + (x_0+y_0)t\mathbf{v}_1]. \end{aligned}$$

Note that the prefactor of the term $e^{\lambda t}t\mathbf{v}_1$ (x_0+y_0) which (since $\mathbf{v}_1 = (1,-1)$) means that it vanishes since $\mathbf{x}_0$ is along $\mathbf{v}_1$. The term $\sim te^{\lambda t}$ therefore only present if $\mathbf{x}_0$ is not parallel to $\mathbf{v}_1$.