9. Lorenz equations and chaos

Three-dimensional ODE with three parameters: σ, r, $b > 0$,

$$\begin{aligned}\dot{x} &= \sigma(y - x),\\ \dot{y} &= rx - y - xz,\\ \dot{z} &= xy - bz.\end{aligned}$$

In 1963 Ed Lorenz found that this deterministic system has extremely erratic behaviour.

The behavior was found to be "chaotic"—depends sensitively on the initial conditions:
Compare $\mathbf{x}(t)$ with $\mathbf{x}'(t)$ starting from $\mathbf{x}_0$ and $\mathbf{x}_0 + \delta_0$, respectively.

Then this small distance grows exponentially:

$$||\mathbf{x}'(t) - \mathbf{x}(t)|| \sim ||\delta_0|| e^{\lambda t}.$$

10. One-dimensional maps

Now: discrete time rather than continuous.

These systems are known as

- difference equations,
- recursion relations,
- iterated maps, or just
- **maps**.

Maps appear in different contexts:

1. For analyzing differential equations.
2. To model natural phenomena.
3. As simple examples of chaos.

Discrete population model for a single species

Many species have no overlap between successive generations, population growth take place through discrete steps:

$$N_{t+1} = N_t F(N_t) = f(N_t).$$

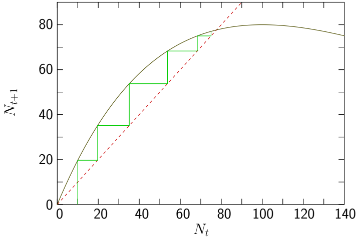
The simplest case: suppose that $F(N_t) = r > 0$:

$$N_{t+1} = rN_t \quad \Rightarrow \quad N_t = r^t N_0. \tag{1}$$

This is usually not very realistic, though it could work for the early stages of growth of certain bacteria.

Graphical solution

Given the function $f(N_t)$ and the starting value N_0 the sequence N_1, N_2, ..., may be determined graphically: The fixed point N^* are intersections of the curve $N_{t+1} = f(N_t)$ and the straight line $N_{t+1} = N_t$.

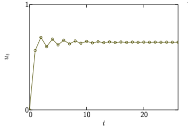


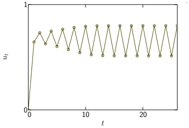
If N^* is a stable or unstable solution is determined by $f'(N_t)$. Examine the effect of a small perturbation from the fixed point:

$$f(N^* + \delta) = N^* + \delta f'(N^*).$$

A small perturbation ...

- will lead to oscillations if $f'(N^*) < 0$,
- will die out if $|f'(N^*)| < 1$.





Discrete population model with a limiting carrying capacity

One generally expects the function $f(N_t)$ to have a maximum at N_m and decrease for $N_t > N_m$. One such model is

$$N_{t+1} = rN_t \left(1 - \frac{N_t}{K}\right),$$

but has the drawback that $N_{t+1} < 0$ if $N_t > K$, which is of course not realistic.

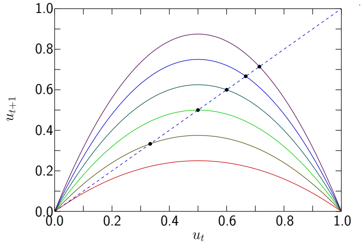
Discrete population model—the logistic model

Rescale with $u_t = N_t/K$ and examine the behavior of

$$u_{t+1} = ru_t(1 - u_t), \quad r > 0.$$

We are interested in solutions with $u_t > 0$, only. The steady states and the corresponding eigenvalues $\lambda = f'(u^*)$ are

$$u_1^* = 0, \qquad \lambda_1 = r,$$
$$u_2^* = \frac{r-1}{r}, \qquad \lambda_2 = 2 - r.$$

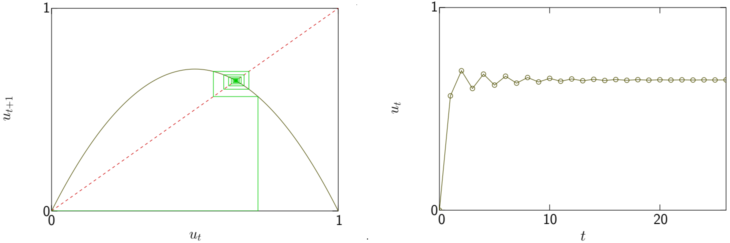


Curves for $r = 1, 1.5, \ldots 3.5$. The solid dots show the respective steady state solutions, u_2^*.

$$\left[\text{Fixed point: } u = ru(1-u) \Rightarrow 1 = r(1-u) \Rightarrow u = 1 - \frac{1}{r}.\right]$$

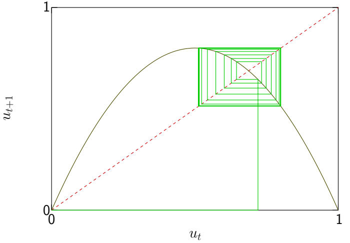
Stable steady state

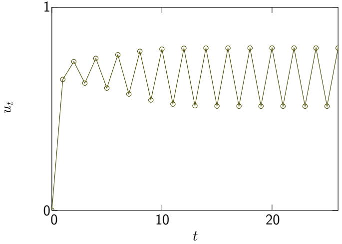
The determining factor for the stability is the eigenvalue, $\lambda = f'(u_2^*) = 2 - r$; the system is stable for $|\lambda| < 1$. The figures here show the behavior for $r = 2.8$ and the initial value $u_0 = 0.72$. The successive iterations take u_t towards the stable solution $u^* = (r-1)/r \approx 0.643$.



Unstable steady state

The value $r = 3.2$ on the other hand gives $\lambda = -1.2$ and we expect the solution $u_2^* \approx 0.688$ to be unstable. Starting at $u_0 = 0.72$, which is close to $u_2^* = (r-1)/r \approx 0.688$, we see, just as expected, that successive iterations take us away from the steady state. As the figures below show this gives a new behavior with an oscillation with period=2.

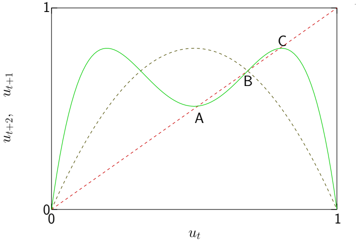




Period doubling

To examine this behavior in more detail we consider the map from u_t to u_{t+2} defined by

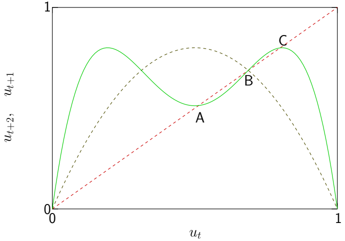
$$\begin{aligned} u_{t+1} &= r u_t(1-u_t), \\ u_{t+2} &= r u_{t+1}(1-u_{t+1}). \end{aligned}$$



The figure to the right is for $r = 3.2$ and we see that there are then three solutions to $u_{t+2} = u_t$, which we denote by u_A, u_B, and u_C. Of these u_B is an unstable solution since it has $f'(u_B) > 1$. (Note that $u_B = u_2^*$). On the other hand, u_A and u_C are stable.

Period doubling. . . cont'd

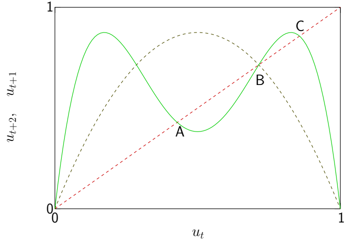
Note what this means: $u_t = u_A$ will give $u_{t+2} = u_A$, whereas $u_t = u_C$ similarly gives $u_{t+2} = u_C$. Together with the earlier figure we may conclude that that system oscillates, u_A, u_C, u_A, u_C, ...

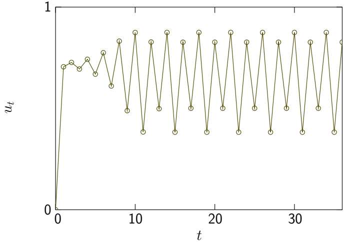


This phenomenon—which is here seen as a change from a simple steady state to an oscillation between two different values—is called a *bifurcation*.

Period doubling. . . cont'd

The discussion above may now be taken one more step in the same direction as shown in this figure, which is for $r = 3.5$: when r increases the solutions u_A and u_C move somewhat, and the steady states eventually become unstable as $f'(u_C) < -1$ and $f'(u_A) < -1$.





When that happens we again get a *period doubling* and the system repeats itself after 4 units of time.

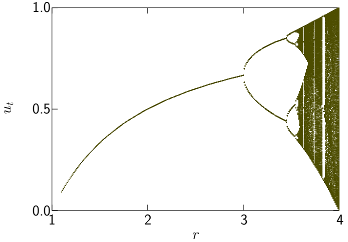
Period doubling. . . cont'd

It is then possible to repeat the discussion in this section for u_{t+4}. We would then find four solutions to $u_{t+4} = u_t$ which are stable for $r = 3.5$ but turn unstable at a slightly larger r. This instability gives an oscillation of period eight, and it now seems that this period doubling may be continued without limit.

It turns out that the distance in r between successive bifurcations decreases rapidly, and as we approach $r_c \approx 3.5699456$ the oscillations have period 2^k with $k \to \infty$. For $r > r_c$ the behavior is aperiodic and we there enter the region of chaos.

Chaos

It turns out that the behavior changes in unexpected ways as a function of r:



- For $1 < r \leq 3$ there is a unique solution $(r-1)/r$.
- For $3 < r \leq 1+\sqrt{6}(\approx 3.45)$ the system has periodic fluctuations between two values.
- For $1+\sqrt{6} < r < 3.54$ (approximately) the system has periodic oscillations between four values.
- For $3.54 < r < 3.57$ the system oscillates between 8, 16, 32, values, etc.
- At $r \approx 3.57$ is the onset of chaos. We can no longer see any oscillations of finite period and slight variations in the initial value yields dramatically different results over time.