

New and old problems

1 Indoor air humidity

Since the cold outdoor air contains a very small amount of water the indoor air becomes very dry during winter. Assume an outdoor temperature of 0°C and 100% relative humidity and indoor temperature of 20°C .

- a) Calculate the relative humidity of the indoor air. Assume that the outdoor air is quickly circulating to the inside, as is common of modern buildings.
- b) Sara wants to rise the humidity in her apartment of 23 m^2 and 2.4 m from floor to ceiling by 50 percentage (which could e.g. be from 5% to 55%) by boiling water. How much water needs to be boiled away? How much energy is needed in this process?

Solution:

- a) Partial pressures: 0.00611 and 0.023, respectively. The relative humidity becomes 26.6%.
- b) We have $0.50 \times P_{\text{H}_2\text{O}}V = nRT$ gives $n = 0.5 \times 0.023 \times 10^5 \times 23 \times 2.4 / 8.31 / 293 = 26$ moles which is $26 \times 18\text{g} = 0.468\text{ kg}$. This means an energy of $Lm = 2260 \times 10^3 \times 0.468 = 1.057\text{MJ}$.

Answer:

- a) The relative humidity becomes 26.6%.
- b) To increase the relative humidity by 50% we need to boil away 0.468 kg which requires 1.06 MJ of energy.

2 Opening refrigerator door

You open a kitchen refrigerator and close it again. In this process the volume fraction $\epsilon = 0.1$ of the cold air in the refrigerator is replaced by warm air of the room. After a short pause you try to open the refrigerator again and find that you then need to apply a large force. Calculate this force!

Take the room temperature to be $T_1 = 20^{\circ}\text{C}$, the temperature in the refrigerator to be $T_2 = 0^{\circ}\text{C}$ and the refrigerator door to have an area of $A = 1\text{m}^2$.

Solution: Introduce refrigerator temperature, $T_2 = 278$ K, room temperature $T_1 = 293$ K. Before opening the door we have

$$PV = Nk_B T_2.$$

After opening and closing the door the volume fraction ϵ is replaced with warm air. The number of warm and cool molecules is now

$$\begin{aligned} P(1 - \epsilon)V &= N_2 k_B T_2, \\ P\epsilon V &= N_1 k_B T_1. \end{aligned}$$

We now use the new number of molecules, $N_1 + N_2$:

$$P_{\text{new}} V = (N_1 + N_2) k_B T_2 = \frac{P\epsilon V}{k_B T_1} k_B T_2 + P(1 - \epsilon)V,$$

which becomes

$$P_{\text{new}} = \left[\epsilon \frac{T_2}{T_1} + (1 - \epsilon) \right] P.$$

Use pressure difference

$$P - P_{\text{new}} = \epsilon \left(1 - \frac{T_2}{T_1} \right) P$$

to determine the force

$$F = A(P - P_{\text{new}}) = 1 \times 0.1 \left(1 - \frac{278}{293} \right) \times 10^5 = 512 \text{ N}.$$

Answer:

$$P_{\text{atm}} A \epsilon \left(1 - \frac{T_2}{T_1} \right)$$

3 Chemical potential

The heat capacity of a degenerate electron gas at constant volume may be written as

$$C_v = aN \left(\frac{V}{N} \right)^{2/3} T,$$

where a is a factor involving only universal constants of physics. Determine how the chemical potential of the gas depends on temperature.

Answer:

$$\mu(T) = -\frac{a}{6} \left(\frac{V}{N} \right)^{2/3} T^2.$$

Basics We can determine S and U from

$$C_v = T \left(\frac{\partial S}{\partial T} \right)_V \Rightarrow S = \int \frac{C_V}{T} dT = aN^{1/3}V^{2/3}T,$$

and

$$C_V = \left(\frac{\partial U}{\partial T} \right)_V \Rightarrow U = \int C_V dT = aN^{1/3}V^{2/3}\frac{T^2}{2}.$$

Incorrect approach One can then try

$$\mu = -T \left(\frac{\partial S}{\partial N} \right)_{U,V} = -\frac{a}{3} \left(\frac{V}{N} \right)^{2/3} T^2,$$

or

$$\mu = \left(\frac{\partial U}{\partial N} \right)_{S,V} = \frac{a}{6} \left(\frac{V}{N} \right)^{2/3} T^2,$$

but they give different answers which both are wrong.

Hint The reason that this is incorrect is that we have not kept U constant in the first case and S constant in the second. To keep U constant when we differentiate S the temperature has to change as N changes. Formally differentiating the expression for U gives

$$dU = a\frac{1}{3}N^{-2/3}V^{2/3}\frac{T^2}{2}dN + aN^{1/3}V^{2/3}TdT,$$

and taking $dU = 0$ leads to

$$\frac{1}{3}N^{-2/3}\frac{T^2}{2}dN = -N^{1/3}dT \Rightarrow dN = -6\frac{N}{T}dT,$$

whereas a similar approach with $dS = 0$ leads to $dN = -3\frac{N}{T}dT$. With this approach there will be two terms in $(\partial S/\partial N)$ and $(\partial U/\partial N)$ and both paths will lead to the correct answer.

We then use

$$dU = \left(\frac{\partial U}{\partial N} \right)_{VT} dN + \left(\frac{\partial U}{\partial T} \right)_{VN} dT,$$

which gives

$$\begin{aligned}
\left(\frac{\partial U}{\partial N}\right)_{SV} &= \left(\frac{\partial U}{\partial N}\right)_{VT} + \left(\frac{\partial U}{\partial T}\right)_{VN} \left(\frac{\partial T}{\partial N}\right)_{SV} \\
&= a \frac{1}{3} N^{-2/3} V^{2/3} \frac{T^2}{2} - a N^{1/3} V^{2/3} T \frac{T}{3N} \\
&= \left(\frac{a}{6} - \frac{a}{3}\right) \left(\frac{V}{N}\right)^{2/3} T^2 \\
&= -\frac{a}{6} \left(\frac{V}{N}\right)^{2/3} T^2
\end{aligned}$$

4 Heat engine

Consider a heat engine with a cycle consisting of three steps:

1. adiabatic compression from V_1 to V_2 ,
2. isothermal expansion back to volume V_1 ,
3. pressure relaxation to the initial state at constant volume.

Find the efficiency of such an engine. Assume that the working substance is a two-atomic gas with “frozen out” oscillation degrees of freedom.

Solution:

1. Adiabatic process: no heat. With $PV^\gamma = \text{const}$ which gives $TV^{\gamma-1} = \text{const}$ we have $T_1 V_1^{\gamma-1} = T_2 V_2^{\gamma-1}$.
2. Isothermal: $\Delta U = 0$, $Q_2 = \Delta U - W$, $W = -\int P dV = -Nk_B T_2 \int_{V_2}^{V_1} \frac{dV}{V} = -Nk_B T_2 \ln(V_1/V_2)$, gives $Q_h = Q_2 = Nk_B T_2 \ln(V_1/V_2)$
3. Isochoric: $W = 0$, $Q = \Delta U = \frac{f}{2} Nk_B (T_1 - T_2)$, $Q_c = -Q = \frac{f}{2} Nk_B (T_2 - T_1)$.

Taken together:

$$\eta = 1 - \frac{Q_c}{Q_h} = 1 - \frac{f}{2} \frac{Nk_B (T_2 - T_1)}{Nk_B T_2 \ln(V_1/V_2)} = 1 - \frac{f}{2} \frac{1 - (V_2/V_1)^{\gamma-1}}{\ln(V_1/V_2)}.$$

Answer:

$$\eta = 1 - \frac{f}{2} \frac{1 - (V_2/V_1)^{\gamma-1}}{\ln(V_1/V_2)}.$$

5 Phase transition in ^3He

Consider the phase transition between solid and liquid ^3He at low temperatures. At absolute zero the transition happens at a pressure P_0 , the solid phase corresponds to $P > P_0$, the liquid phase is at $P < P_0$. The entropy per mole of the liquid phase is given by $S_l = \alpha T$ whereas the entropy of the solid phase may be taken to be a constant $S_s = S_0$. The volume per mole of both the liquid and the solid phases may be taken to be a constant with $\Delta V = V_l - V_s = V_0 > 0$.

Determine the phase boundary curve and the critical pressure, P_c , below which solid ^3He cannot exist. *Hint:* Consider Fig. 5.13 in the textbook.

Solution: Use $\Delta S = \alpha T - S_0$ in

$$\frac{dP}{dT} = \frac{\Delta S}{\Delta V} = \frac{\alpha T - S_0}{V_0},$$

which gives

$$P(T) = P_0 + \int_0^T \frac{dP}{dT} dT' = P_0 + \frac{1}{V_0} \left(\alpha \frac{T^2}{2} - S_0 T \right).$$

Plug in the highest temperature with a non-zero ΔS , i.e. $T = S_0/\alpha$:

$$P_c = P_0 + \frac{1}{V_0} \left(\alpha \frac{S_0^2}{2\alpha^2} - \frac{S_0^2}{\alpha} \right) = P_0 - \frac{S_0^2}{2V_0\alpha}.$$

Answer:

$$P_c = P_0 - \frac{S_0^2}{2V_0\alpha}.$$

6 Moving mass *Note that this is an unusually difficult problem.*

Consider a thermally isolated chamber of total volume $2V$ filled with a one-atomic gas with initial temperature T_i . The volume is separated into two parts by a movable plate of surface area A and an unknown mass M as shown in the figure. The plate allows them to interact between the upper and the lower parts of the chamber. We have the same amount of particles N in each part. Initially the plate is fixed

and the volumes of both parts are the same and equal to V . We then let the plate move under the action of gravity. After a transitional period the system reaches mechanical and thermal equilibrium with a new temperature of the gas and new volumes, $V_1 = V/2$ for the lower part and $V_2 = 3V/2$ for the upper part. Find the mass M of the plate. Neglect the potential energy of the gas. *Hint: Think about energy conservation in the system.*

Answer:

$$Mg = \frac{12}{7} \frac{Nk_B T_i A}{V}$$

7 Entropy change

Consider a chamber of total volume $2V$ separated into two parts of volume $V_1 = V/2$ and $V_2 = 3V/2$. The parts are filled with one-atomic gases A and B , respectively, which are initially in mechanical, but not thermal equilibrium with each other. There is an equal amount of atoms for each gas, $N_A = N_B = N$. We then remove the plate and thus allow thermal and diffusive interaction between gases. Find the increase in entropy of the system.

Answer:

$$\Delta S = nk_B \ln \left(\frac{128}{27} \sqrt{3} \right)$$

8 Chemical potential of a crystal

According to the Debye model, the heat capacity of a crystal at low temperatures may be written as

$$C_v = aNk_B \left(\frac{T}{T_D} \right)^3,$$

where a is a numerical factor and T_D is a constant known as the Debye temperature. Find how the chemical potential of the crystal depends on the temperature.

Answer:

$$\mu = -\frac{ak_B T^4}{12T_D^3}$$

9 Engine

Consider an engine cycle with three steps,

1. adiabatic compression from V_1 to V_2 ,
2. isobaric expansion back to volume V_1 ,
3. pressure relaxation to the initial state at constant volume.

Find the efficiency of such an engine. Assume that the working substance is a two-atomic gas with “frozen out” oscillation degrees of freedom.

Solution: (Corrected 2020-01-11, 13:40)

1. Adiabatic process: no heat. $PV^\gamma = \text{const}$ gives $P_1 V_1^\gamma = P_2 V_2^\gamma$.
2. Isobaric expansion back to V_1 and temperature T_3 : $W = -P_2(V_1 - V_2)$, $\Delta U = \frac{f}{2} N k_B (T_3 - T_2) = \frac{f}{2} P_2 (V_1 - V_2)$
 $Q_h = Q_2 = \Delta U - W = (\frac{f}{2} + 1) P_2 (V_1 - V_2)$.
3. Isochoric: $W = 0$, $Q = \Delta U = \frac{f}{2} (P_1 - P_2) V_1$. $Q_c = -Q = \frac{f}{2} (P_2 - P_1) V_1$.

Taken together:

$$\eta = 1 - \frac{Q_c}{Q_h} = 1 - \frac{f/2}{f/2 + 1} \frac{V_1(P_2 - P_1)}{P_2(V_1 - V_2)} = 1 - \frac{f/2}{f/2 + 1} \frac{1 - P_1/P_2}{1 - V_2/V_1}$$

$$\eta = 1 - \frac{f/2}{f/2 + 1} \frac{1 - (V_2/V_1)^\gamma}{1 - V_2/V_1}$$

$$\eta = 1 - \frac{f/2}{f/2 + 1} \frac{1 - (V_2/V_1)^\gamma}{1 - V_2/V_1}.$$

10 Modified van der Waal's model

Imagine the modified van der Waal's model

$$\left(P + a \frac{N^3}{V^3} \right) (V - bN) = N k_B T,$$

where a and b are some constants. Consider an expansion of such a gas from $V = 5bN$ to $V = 9bN$ at constant N and T and determine the change in Gibbs free energy of the gas.

Answer: My result

$$\Delta G = -Nk_B \ln 2 - \frac{Nk_B T}{8} + \frac{28aN}{27 \cdot 25b^2}.$$

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$$\Delta G = -Nk_B \ln 2 - \frac{Nk_B T}{8Nb} + \frac{9aN}{128b^2}.$$

11 Leaking container

A container of volume V is filled with a gas of temperature T , initial total number of particles N_0 , and mass m for each particle. We make an extremely small hole of area A in the container and the gas starts leaking out slowly due to random collisions of gas particles with the wall (or rather, the hole). How does the number of gas particles N decrease with time?

Hint: Find how many particles dN that pass through the hole in a time interval dt and solve the equation for dN/dt . Take the average thermal velocity in one direction to be $\langle v_x \rangle \approx \langle v_x^2 \rangle^{1/2}$.

Answer:

$$N = N_0 e^{-t/\tau}, \quad \tau = 2 \frac{V}{A} \sqrt{\frac{m}{k_B T}}.$$

Solution: The number of particles that leave the container is the same as the number of particles in the volume $A \times \langle v_x \rangle dt/2$. The division by 2 is because only half of the particles will move towards the hole. The number of particles is then

$$dN = -\frac{N}{V} A \langle v_x \rangle dt/2 \quad \Rightarrow \quad \frac{dN}{dt} = -N \frac{A \langle v_x \rangle}{2V} = -N/\tau.$$

The solution to this is

$$N = N_0 e^{-t/\tau},$$

and since $m \langle v_x^2 \rangle = k_B T$, we have

$$\tau = \frac{2V}{A} \sqrt{\frac{m}{k_B T}}.$$

12 Heat capacity

Imagine a certain material with heat capacity given by $C_v = aT^3$, where a is some constant. Find how the entropy of this material depends on the energy, U .

Answer:

$$S = \frac{1}{3}a^{1/4}(4U)^{3/4}.$$