Problem 4.20

Find the efficiency of the Diesel cycle!
Solution by V. Bychkov. Notations from Fig. 4.6.

There are four steps:

$1 \to 2$ adiabatic compression,

$2 \to 3$ ignition at constant pressure, P_2,

$3 \to 4$ power stroke—adiabatic expansion,

$4 \to 1$ exhaust at constant volume.

Solution

$$e = 1 - \frac{Q_c}{Q_h}$$

The energy of an ideal gas:

$$U = \frac{f}{2} N k_B T = \frac{f}{2} PV. \tag{1}$$

From the heat flowing into the system

$$\begin{aligned} Q_h &= Q_{2\to 3} = \Delta U_{2\to 3} - W_{2\to 3} = U_3 - U_2 + \int_2^3 P_2\, dV \\ &= \frac{f}{2} P_2(V_3 - V_2) + P_2(V_3 - V_2) = \left(\frac{f}{2} + 1\right) P_2(V_3 - V_2). \end{aligned}$$

and the heat flowing out (note the sign convention)

$$Q_c = -Q_{4\to 1} = -\Delta U_{4\to 1} = \frac{f}{2} V_1(P_4 - P_1),$$

we get (with $\gamma = (f+2)/f$)

$$\begin{aligned} \frac{Q_c}{Q_h} &= \frac{\frac{f}{2} V_1(P_4 - P_1)}{\left(\frac{f}{2} + 1\right) P_2(V_3 - V_2)} \\ &= \frac{V_1(P_4 - P_1)}{\gamma P_2(V_3 - V_2)}. \end{aligned}$$

We then make use of the relations for the adiabatic steps:

$$P_4 V_1^\gamma = P_2 V_3^\gamma \quad \Rightarrow \quad P_4 = P_2 \left(\frac{V_3}{V_1}\right)^\gamma,$$

and

$$P_1 V_1^\gamma = P_2 V_2^\gamma \quad \Rightarrow \quad P_1 = P_2 \left(\frac{V_2}{V_1}\right)^\gamma,$$

to get

$$\begin{aligned}\frac{Q_c}{Q_h} &= \frac{V_1}{\gamma(V_3 - V_2)} \left[\left(\frac{V_3}{V_1}\right)^\gamma - \left(\frac{V_2}{V_1}\right)^\gamma\right] \\ &= \frac{1}{\gamma} \frac{V_1/V_2}{V_3/V_2 - 1} \left(\frac{V_2}{V_1}\right)^\gamma \left[\left(\frac{V_3}{V_2}\right)^\gamma - 1\right].\end{aligned}$$

The efficiency then becomes

$$e = 1 - \frac{1}{\gamma} \left(\frac{V_2}{V_1}\right)^{\gamma - 1} \frac{(V_3/V_2)^\gamma - 1}{V_3/V_2 - 1}.$$

For compression ratio $V_1/V_2 = 18$ and cutoff ration $V_3/V_2 = 2$ the efficiency becomes $e = 63\%$.

Less efficient than the Otto cycle

For a given compression ratio the Diesel cycle is less efficient than the Otto cycle. Show that!

Consider first V_3 just slightly larger than V_2, $V_3/V_2 = 1 + \alpha$. We then keep only the first term in the Taylor expansion

$$(1 + \alpha)^\gamma \approx 1 + \gamma\alpha,$$

and the efficiency becomes

$$e \approx 1 - \left(\frac{V_2}{V_1}\right)^{\gamma - 1} \frac{1}{\gamma} \frac{(1 + \gamma\alpha) - 1}{1 + \alpha - 1} = 1 - \left(\frac{V_2}{V_1}\right)^{\gamma - 1},$$

which agrees with the formula for the Otto cycle. When keeping more terms in the Taylor expansion of $(1 + \alpha)^\gamma$, the negative term becomes bigger and it is then clear that $e_{\text{Diesel}} < e_{\text{Otto}}$.