Problem 2.24

Part (a)

The height of the peak is when $N_\uparrow = N_\downarrow = N/2$,

$$\Omega_{\text{max}} = \frac{N!}{(N/2)!(N/2)!}.$$

We will now use the Stirling approximation, $\ln x! \approx x \ln x - x$, which may also be written

$$x! \approx \left(\frac{x}{e}\right)^x,$$

but we will also see that this is a situation where we should rather use the more precise Stirling approximation.

We get

$$\Omega \approx \frac{(N/e)^N}{[(N/2e)^{N/2}]^2} = \frac{(N/e)^N}{(N/e)^N(1/2)^N} = 2^N$$

This can however not be correct since 2^N is the total number of states for the two-state paramagnet, and the reason for this is that this simple Stirling approximation is not very accurate.

Better solution!

The more careful solution instead makes use of

$$x! \approx \left(\frac{x}{e}\right)^x \sqrt{2\pi x},$$

and we then find

$$\Omega_{\text{max}} \approx \frac{(N/e)^N\sqrt{2\pi N}}{[(N/2e)^{N/2}\sqrt{\pi N}]^2} = \frac{(N/e)^N}{(N/e)^N(1/2)^N}\frac{\sqrt{2\pi N}}{\pi N} = \frac{2^N}{\sqrt{\pi N/2}}.$$

Part (b)

We have

$$\Omega = \frac{N!}{N_\uparrow!(N-N_\uparrow)!}$$

Take $N_\uparrow! = N/2 + x$, $N - N_\uparrow = N/2 - x$, and use the simpler Stirling approximation:

$$\begin{aligned}
\ln\Omega &\approx N\ln N - N - [(N/2+x)\ln(N/2+x) - (N/2+x)] \\
&\quad -[(N/2-x)\ln(N/2-x) - (N/2-x)] \\
&= N\ln N - (N/2+x)\ln\left[\frac{N}{2}\left(1+\frac{2x}{N}\right)\right] - (N/2-x)\ln\left[\frac{N}{2}\left(1-\frac{2x}{N}\right)\right].
\end{aligned}$$

We then use $\ln\left[\frac{N}{2}\left(1+\frac{2x}{N}\right)\right] = \ln\frac{N}{2} + \ln\left(1+\frac{2x}{N}\right)$ and the Taylor expansion to second order for the second term,

$$\ln(1+\epsilon) \approx \epsilon - \frac{1}{2}\epsilon^2 + \dots,$$

which is necessary since we the term linear in x vanishes.

$$\begin{aligned}
\ln\Omega &\approx N\ln N - (N/2+x)\left[\ln\frac{N}{2} + \left(\frac{2x}{N}\right) - \frac{1}{2}\left(\frac{2x}{N}\right)^2\right] \\
&\quad -(N/2-x)\left[\ln\frac{N}{2} + \left(-\frac{2x}{N}\right) - \frac{1}{2}\left(-\frac{2x}{N}\right)^2\right] \\
&= N\ln N - N\left[\ln\frac{N}{2} - \frac{1}{2}\left(\frac{2x}{N}\right)^2\right] - x\left[2\left(\frac{2x}{N}\right)\right] \\
&= N\ln 2 + \frac{2x^2}{N} - \frac{4x^2}{N} \\
&= N\ln 2 - \frac{2x^2}{N},
\end{aligned}$$

which gives

$$\Omega(x) = 2^N e^{-2x^2/N}.$$

With the better Stirling approximation that instead becomes

$$\Omega(x) = \Omega_{\max}\ \exp\left(-\frac{x^2}{2(N/4)}\right).$$

Incorrect solution

With only the first term in the Taylor expansion $\ln(1+\epsilon) \approx \epsilon$ we arrive at

$$\begin{aligned}
\ln \Omega &\approx N \ln N - (N/2 + x)\left[\ln \frac{N}{2} + \left(\frac{2x}{N}\right)\right] - (N/2 - x)\left[\ln \frac{N}{2} + \left(-\frac{2x}{N}\right)\right] \\
&= N \ln N - N\left[\ln \frac{N}{2}\right] - 2x\left(\frac{2x}{N}\right) \\
&= N \ln 2 - \frac{4x^2}{N},
\end{aligned}$$

which is not correct.

Alternative approach

One can instead use the fact that $\ln(a+b) + \ln(a-b) = \ln(a^2 - b^2)$. It is then enough to use the first order Taylor expansion, since the first term in the expansion gives the x^2 term.

$$\begin{aligned}
\ln \Omega &\approx N \ln N - (N/2 + x)\ln(N/2 + x) - (N/2 - x)\ln(N/2 - x) \\
&= N \ln N - \frac{N}{2}\left[\ln\left(\frac{N}{2} + x\right) + \ln\left(\frac{N}{2} - x\right)\right] - x\left[\ln\left(\frac{N}{2} + x\right) - \ln\left(\frac{N}{2} - x\right)\right] \\
&= N \ln N - \frac{N}{2}\ln\left(\frac{N^2}{4} - x^2\right) - x\left[\ln\left(\frac{N}{2} + x\right) - \ln\left(\frac{N}{2} - x\right)\right] \\
&= N \ln N - \frac{N}{2}\ln\left[\frac{N^2}{4}\left(1 - \frac{4x^2}{N^2}\right)\right] - x\left[\ln\left(\frac{N}{2} + x\right) - \ln\left(\frac{N}{2} - x\right)\right] \\
&= N \ln N - \frac{N}{2}\left[\ln \frac{N^2}{4} - \frac{4x^2}{N^2}\right] - x\left[\left(\ln \frac{N}{2} + \frac{2x}{N}\right) - \left(\ln \frac{N}{2} - \frac{2x}{N}\right)\right] \\
&= N \ln N - \frac{N}{2}\ln\left(\frac{N}{2}\right)^2 + \frac{2x^2}{N} - \frac{4x^2}{N} \\
&= N \ln 2 - \frac{2x^2}{N},
\end{aligned}$$

which (again with the better Stirling approximation) gives

$$\Omega(x) = \Omega_{\max}\ \exp\left(-\frac{x^2}{2(N/4)}\right).$$

Sum of $\Omega(x)$

We expect the sum over all the multiplicities should give

$$\sum_{x=-N/2}^{N/2} \Omega(x) = 2^N.$$

We then approximate the sum by the integral and make use of

$$\int e^{-x^2/2/a^2} = \sqrt{2\pi}a,$$

to get

$$\int dx\ \Omega(x) = \Omega_{\max} \int dx\ e^{-x^2/2/(N/4)} = 2^N \sqrt{\frac{2}{N\pi}} \sqrt{2\pi} \sqrt{\frac{N}{4}} = 2^N.$$

That this comes out exactly correct doesn't show that the calculations are exact. We are using approximations in many places, e.g. by approximating the sum by an integral. The conclusion is rather that the different approximations have the same origin such that they cancel each other out.