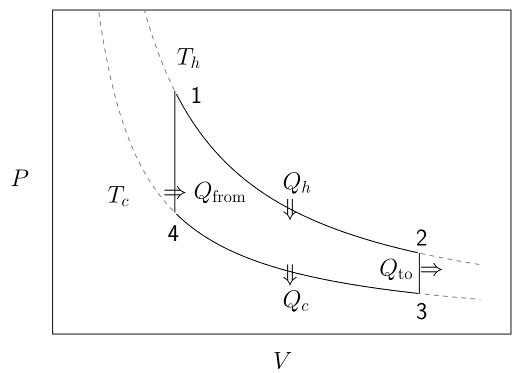
Problem 4.21

The ingenious **Stirling engine** is a true heat engine that absorbs heat from an external source. See p. 133–134 for a description of its workings.

(a) Draw a PV diagram for this idealized Stirling cycle.

The cycle has two isotherms and two isochores. Q_{from} and Q_{to} are the heat flows from and to the regenerator.



We need to express the heat flows in terms of the Q that follow the sign convention:

$$
\begin{aligned}
Q_h &= Q_{12}, \\
Q_{\text{to}} &= -Q_{23}, \\
Q_c &= -Q_{34}, \\
Q_{\text{from}} &= Q_{41}.
\end{aligned}
$$

For the work done by the engine we have

$$W_e = -W.$$

We then consider the four different stages

$1 \to 2$ An isothermal process with temperature $T = T_h$. The work is

$$W_{12} = -\int_{V_1}^{V_2} P(V)dV = -Nk_BT_h \int \frac{dV}{V} = -Nk_BT_h \ln \frac{V_2}{V_1}.$$

With $\Delta U = 0$ and $\Delta U = Q + W$ we get

$$Q_{12} = -W_{12} = Nk_BT_h \ln \frac{V_2}{V_1}.$$

$2 \to 3$ Isochoric process (constant volume) with $W_{23} = 0$.
We have

$$Q_{23} = \Delta U_{23} = \frac{f}{2} N k_B (T_c - T_h) < 0,$$

which means that heat is leaving the system.

$3 \to 4$ Isothermal, $T = T_c$:

$$W_{34} = -\int_{V_3}^{V_4} P(V) dV = -N k_B T_c \ln \frac{V_4}{V_3} = N k_B T_c \ln \frac{V_2}{V_1}.$$

Since $\Delta U_{34} = 0$ the heat becomes

$$Q_{34} = -W_{34} = -N k_B T_c \ln \frac{V_2}{V_1} < 0.$$

$4 \to 1$ Isochoric process, $W_{41} = 0$.

$$Q_{41} = \Delta U_{41} = \frac{f}{2} N k_B (T_h - T_c) > 0,$$

which means that heat is entering the system.

(b) Forget about the regenerator for the moment. Then, during step 2, the gas will give up heat to the cold reservoir instead of to the regenerator; during step 4 the gas will absorb heat from the hot reservoir. Calculate the efficiency of the engine in this case, assuming that the gas is ideal. Express your answer in terms of the temperature ratio T_c/T_h and the compression ratio (the ratio of the maximum and minimum volumes). Show that the effficiency is less than that of a Carnot engine operating between the same temperatures. Work out a numerical example.

We are here asked to consider the total quantities

$$\begin{aligned} Q_h^{\text{tot}} &= Q_h + Q_{\text{from}}, \\ Q_c^{\text{tot}} &= Q_c + Q_{\text{to}}, \end{aligned}$$

and the efficiency becomes $e = 1 - Q_c^{\text{tot}}/Q_h^{\text{tot}}$. We have

$$\frac{Q_c^{\text{tot}}}{Q_h^{\text{tot}}} = \frac{Q_{12} + Q_{41}}{-Q_{34} - Q_{23}} = \frac{T_c \ln \frac{V_2}{V_1} + \frac{f}{2}(T_h - T_c)}{T_h \ln \frac{V_2}{V_1} + \frac{f}{2}(T_h - T_c)} = \frac{T_c}{T_h} \frac{\ln \frac{V_2}{V_1} + \frac{T_h}{T_c}\frac{f}{2}(1 - \frac{T_c}{T_h})}{\ln \frac{V_2}{V_1} + \frac{f}{2}(1 - \frac{T_c}{T_h})} > \frac{T_c}{T_h},$$

which means that $e < 1 - T_c/T_h$, which is thus lower than the Carnot efficiency..

(c) Now put the regenerator back. Argue that, if it works perfectly, the efficiency of a Stirling engine is the same as that of a Carnot engine.

Since $Q_{\mathrm{from}} = Q_{\mathrm{to}}$ there is not net flow of heat from the regenerator. We then have

$$e = 1 - \frac{Q_c}{Q_h} = 1 - \frac{Q_{12}}{-Q_{34}} = 1 - \frac{Nk_BT_h \ln\frac{V_2}{V_1}}{Nk_BT_c \ln\frac{V_2}{V_1}} = 1 - \frac{T_c}{T_h}.$$

An alternative solution is to use W_e and Q_h:

$$e = \frac{W_e}{Q_h} = \frac{-W_{12} - W_{34}}{Q_{12}} = \frac{Nk_BT_h \ln\frac{V_2}{V_1} - Nk_BT_c \ln\frac{V_2}{V_1}}{Nk_BT_h \ln\frac{V_2}{V_1}} = 1 - \frac{T_c}{T_h}.$$

(d) Discuss, in some detail, the various advantages and disadvantages of a Stirling engine, compared to other engines.

The engine produces a very low power. Since the temperature changes along the regenerator the temperature difference between the gas and the regenerator is at each point quite small, which means that not much entropy is produced. A drawback is that it requires a complicated mechanical device that makes it possible for one of the pistons to move while the other is fixed.