1 Jamming as a critical phenomenon

We here consider slowing down of the dynamics when the system becomes too crowded—compare traffic jam. What we have in mind here is collections of particles that are sufficiently big that the thermal fluctuations don't play any role. We therefore do simulations with temperature $T = 0$. We also mostly do the simulations in 2D. (This case is exceptional in the sense that the critical exponents appear to be the same in 2D and 3D.)

One way to approach the jamming transition is to look at static packings: Throw out the particles at random positions in the simulation cell. Let the particles move according to the repulsive contact forces (see below) and check if one reaches a state without any overlaps. Otherwise the configuration is said to be "jammed".

The second approach—the one that we will consider here—is to apply a dynamics by shearing the system. We are thus sampling configurations from steady state. There is no thermal equilibrium.

1.1 Simplest model for shear-driven jamming

We consider elastic particles with contact-only interaction; the particles interact when the distance $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ is smaller than the sum of their radii, $d_{ij} = \frac{1}{2}(d_i + d_j)$. The force from particle j on particle i is

$$\mathbf{f}_{ij}^{\mathrm{el}} = \frac{k_e}{d_{ij}} \left(1 - \frac{r_{ij}}{d_{ij}}\right) \hat{\mathbf{r}}_{ij}, \tag{1}$$

and the total force becomes

$$\mathbf{f}_i^{\mathrm{el}} = \sum_j \mathbf{f}_{ij}^{\mathrm{el}}, \tag{2}$$

where the sum is over all particles j in contact with particle i.

We further take the particles to sit on some kind of shearing substrate with a y-dependent velocity, $\mathbf{v}_S(\mathbf{r}) = \dot{\gamma} y \hat{x}$. We assume overdamped dynamics (no acceleration, velocity relative to substrate proportional to the force) and get

$$\mathbf{v}_i = \mathbf{v}_S(\mathbf{r}_i) + \frac{1}{k_d} \mathbf{f}_i^{\mathrm{el}}. \tag{3}$$

1.2 Liquid-solid transition

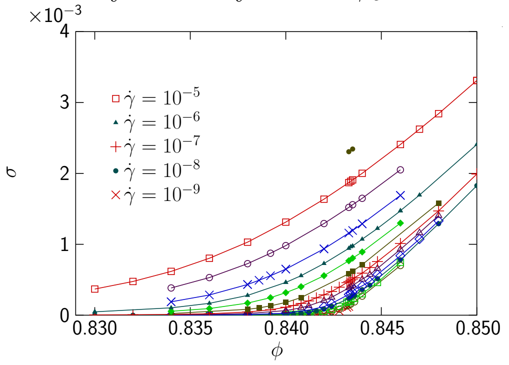
Data from shearing with different shear rates, $\dot{\gamma}$. We determine the shear stress

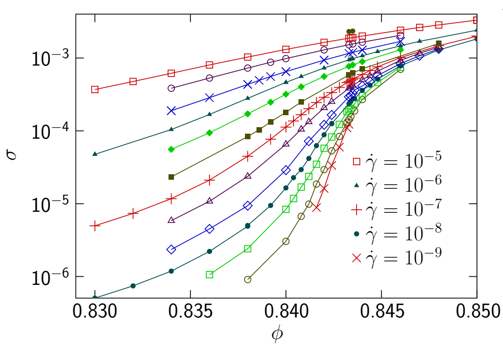
$$\sigma = -\frac{1}{L^2}\sum_{j>i} f_{ij}^x y_{ij}, \tag{4}$$

which is the resistance against shearing. To locate the transition between liquid and solid one should consider the case of slow shearing,

$$\lim_{\dot{\gamma}\to 0} \sigma(\phi, \dot{\gamma}).$$

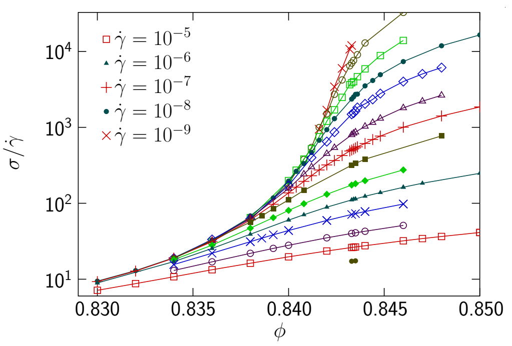
In this way one may locate $\phi_J \approx 0.843$.





1.3 Linear behavior

When the overlap is very small the particles should follow exactly the same paths, only with different velocitites $\propto \dot{\gamma}$. This also means that all the forces should be $\propto \dot{\gamma}$ and with Eq. (4) we conclude that $\sigma/\dot{\gamma}$ should collapse in the low-$\dot{\gamma}$ limit.



1.4 Scaling analysis

In this case we don't have any expression for the free energy, but our starting scaling assumption is for a quantity that is like an order parameter,

$$\sigma(\delta\phi, \dot{\gamma}) = b^{-y/\nu} \tilde{f}(\delta\phi b^{1/\nu}, \dot{\gamma} b^z), \tag{5}$$

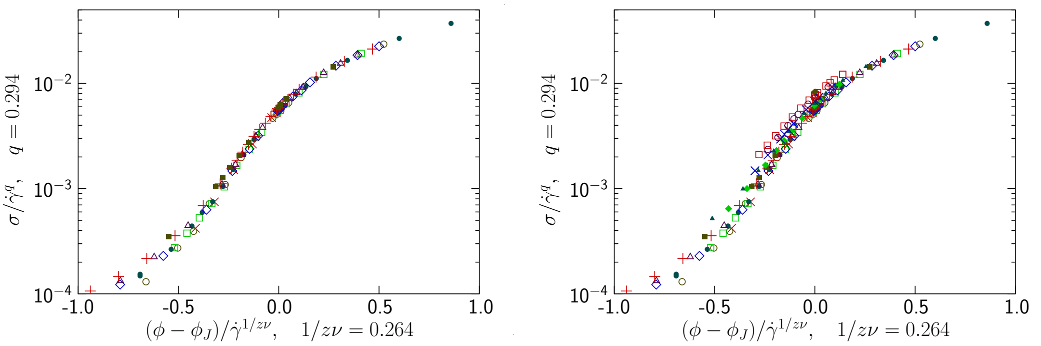
where $\delta\phi = \phi - \phi_J$. Taking $\dot{\gamma} b^z = 1$ gives $b = \dot{\gamma}^{-1/z}$ and

$$\sigma(\delta\phi, \dot{\gamma}) = b^{y/z\nu} f(\delta\phi/\dot{\gamma}^{1/z\nu}). \tag{6}$$

With $q = y/z\nu$ this becomes

$$\frac{\sigma}{\dot{\gamma}^q} = f(\delta\phi/\dot{\gamma}^{1/z\nu}). \tag{7}$$

The left figure below shows a good collapse of our data for shear rates in the range $10^{-9} \le \dot{\gamma} \le 2 \times 10^{-7}$. The figure to the right also includes data up to $\dot{\gamma} = 2 \times 10^{-5}$ and the collapse is then not at all convincing.



1.5 The exponent β

We are interested in the divergence of the shear viscosity,

$$\eta \equiv \frac{\sigma}{\dot{\gamma}} \sim (\phi_J - \phi)^{-\beta}. \tag{8}$$

Starting from Eq. (7), and noting that we may define a new function $f_2(x) = f(x)/|x|^a$, with some arbitrary power a, we chose that exponent such that the $\dot{\gamma}$-dependence cancel out and write

$$\frac{\sigma}{\dot{\gamma}} = \frac{1}{\dot{\gamma}^{1-q}} \frac{\sigma}{\dot{\gamma}^q} = \frac{1}{\dot{\gamma}^{q-1}} \left| \frac{\dot{\gamma}^{1/z\nu}}{\delta\phi} \right|^{(q-1)z\nu} f_2\left(\frac{\delta\phi}{\dot{\gamma}^{1/z\nu}}\right) = |\phi_J - \phi|^{-(q-1)z\nu} f_2\left(\frac{\delta\phi}{\dot{\gamma}^{1/z\nu}}\right). \tag{9}$$

Since we expect the behavior $|\phi_J - \phi|^{-\beta}$ for sufficiently small $\dot\gamma$ we conclude that $f_2(x) \to$ const for large x. A comparison with Eq. (8) gives

$$\beta = (q-1)z\nu, \tag{10}$$

and the numerical values in the figure above give $\beta \approx 2.67$.

2 Other jamming models

2.1 Reformulation of the minimal model

For a more general discussion it is convenient to reformulate the above dynamics by introducing a dissipative force $\mathbf{f}_i^{\rm dis} = -k_d(\mathbf{v}_i - \mathbf{v}_S)$. Newton's equation then gives

$$m_i \frac{d^2\mathbf{r}_i}{dt^2} = \sum_j [\mathbf{f}_{ij}^{\rm el} + \mathbf{f}_{ij}^{\rm dis}], \tag{11}$$

where m_i is the mass of particle i and the sum is over all particles j in contact with particle i. For overdamped dynamics with $m_i = 0$ we recover Eq. (3),

$$0 = \mathbf{f}_i^{\rm el} - k_d(\mathbf{v}_i - \mathbf{v}_S) \quad \Rightarrow \quad \mathbf{v}_i = \mathbf{v}_S + \frac{1}{k_d}\mathbf{f}_i^{\rm el}.$$

2.2 Contact dissipation

If there is only a number of particles and no substrate (or fluid) that the particles can disspate energy against, the above model is not very realistic. A realistic dissipation model is then "contact dissipation",

$$\mathbf{f}_{ij}^{\rm dis} = -k_d(\mathbf{v}_i - \mathbf{v}_j). \tag{12}$$

It turns out to be difficult to simulate this model with $m_i = 0$. The common choice is instead to take the mass to be sufficiently small that we get essentially the same behavior as in the true overdamped dynamics. It is also interesting to consider effects that appear due to the finite particle masses.

2.3 Particle rotations

A closer look at Eq. (12) shows that it is unrealistic in the sense that there is nothing that guarantees torque balance. The next step towards a more realistic model is therefore to introduce moments of inertia and angular velocities and also modify the dissipation equation to read

$$\mathbf{f}_{ij}^{\rm dis} = -k_d(\mathbf{v}_i^c - \mathbf{v}_j^c), \tag{13}$$

where $\mathbf{v}_i^c$ and $\mathbf{v}_j^c$ are now the velocities at the points of contact.

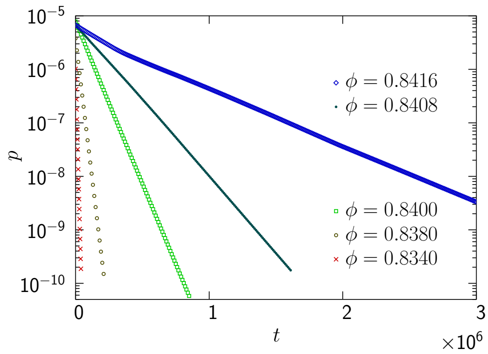
3 Some recent results

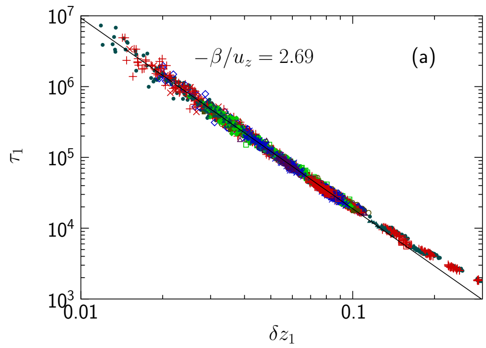
3.1 Contact number and relaxation time

A key result from the research on jamming is that the motion becomes impossible when the particles have an average of four contacts and one therefore expects $\delta z = 4 - z$ to be an important quantity to characterize the system. We will now relate δz to the relaxation time which is determined from a large number of runs where we do the following steps:

1. shear at a constant shear rate,
2. suddenly stop the shearing, and
3. let the system relax to (almost) zero pressure.

At the end of such relaxations the pressure decays exponentially (left figure below) and we can determine the relaxation time τ_1.





By analyzing the final configurations we get the average number of contacts z_1 and thereby values (τ_1, z_1). Plotting τ_1 against $\delta z_1 \equiv 4 - z_1$ give the figure above to the right. With u_z for the vanishing of δz as ϕ_J is approached, $\delta z \sim (\phi_J - \phi)^{u_z}$ and the expectation that τ should behave the same as η together with Eq. (8) we expect $\tau \sim \delta z^{-\beta/u_z}$. With $u_z = 1$ we note that the value $\beta/u_z = 2.69$ is consistent with the value $\beta \approx 2.67$ from above.

3.2 Correlation length

It is possible to determine the correlation length by considering the expectation value of

$$g_x(x) = \langle v_x(\mathbf{r}) v_x(\mathbf{r} + x\hat{x}) \rangle, \tag{14}$$

which is a quantity that (mostly) decays exponentially with a decay length ξ. This ξ diverges as jamming is approached, i.e. $\phi \to \phi_J$ and $\dot{\gamma} \to 0$.

3.3 Non-Newtonian mixtures

A phenomenon of much interest is the dramatic change in the viscosity of mixtures of water and cornstarch from a liquid behavior at slow stirring to a solid-like behavior at more rapid impacts. The common explanation of this phenomenon is that there is usually a liquid layer between the particles that serves to lubricate the contacts but that this layer gets broken when the interparticle forces are above a certain threshold.

This need not be the whole truth. The figure below shows a very dramatic increase in shear viscosity with shear rate in one of our models (contact dissipation plus rotations) even though the model lacks the lubrication layers that are assumed to be essential for this behavior.