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Monte Carlo methods
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Examination, Monte Carlo methods, 7.5hp, 2014–06–04, at 9:00–15:00, Östra paviljongen.

Allowed aids: Calculator, Beta, Physics Handbook.

Hand in each problem on a separate page.
The calculations and the reasoning should be easy to follow.
Good luck!

1 Basic statistics

a) Consider independent, random variables x_i with average μ and variance σ^2. The average of N such variables is $m = (1/N)\sum_i x_i$. What is σ_m^2, the variance of m? (1p)

b) Derive this result! (2p)

c) In simple (as opposed to self-avoiding) random walk we have an important result for $\langle S_N^2 \rangle$ which is related to the above. Here $S_N = \sum_{i=1}^N x_i$ and x_i are independent unit vectors. Show that result, either from basic principles or from the result above. (2p)

d) Specialize to the case where the x_i are from a uniform distribution between -1 and 1 and $N = 100$. What is the distribution of m? (2p)

2 Theory behind Markov chains Monte Carlo

A Markov chain may be described as a transition matrix p_{ij}.

a) Describe the three conditions that have to be fulfilled by this matrix and explain why they are necessary. (2p)
Hint: The first has to be true for all p_{ij}, the second is for p_{ii}, and the last concerns $\sum_j p_{ij}$.

b) Write down the detailed balance condition. Use π_i to denote the desired probability distribution. (2p)

c) Demonstrate that a matrix p_{ij} that fulfills the detailed balance condition will preserve the desired probability distribution π_i. (2p)

3 General questions

a) What is the phenomenon common to earthquakes and other phenomena that self-organized criticality tries to explain? (2p)

b) The Watts-Strogatz model was invented to go gradually from an ordered to a disordered network. Sketch the ordered network and explain the effect of the rewiring probability on the behavior. (1p)

c) Explain *preferential attachment* which was conceived by Barabási and Albert. What is the peculiar property of the networks generated with preferential attachment? (2p)

d) Which of the following statements are true about "universality classes"? (Correct/wrong answers give ±0.5p, but a negative total is taken to zero.) (2p)

1. Two models that are in the same universality class have the same values of all the critical exponents.
2. Two models that are in the same universality class have the same critical temperature.
3. The universality class doesn't change with the number of spin components.
4. The universality class doesn't change with the number of nearest neighbors.

e) Show that the 1D Ising model is disordered for all $T > 0$. (2p)

4 Expectation values

For small lattices it is possible to use three different methods to calculate the properties of the Ising model:

i) Through a complete enumeration of all the possible states.

ii) By generating a set of randomly produced configurations.

iii) Through a Monte Carlo simulation.

a) Describe the formulas that should be used to calculate expectation values in methods i) through iii). Assume that we have access to A_ν and the energy E_ν for each generated configuration and want to calculate $\langle A \rangle$ at inverse temperature β. (3p)

b) Why could it sometimes be motivated to study a system through a complete enumeration? (1p)

c) Consider a complete enumeration of a $L \times L$ Ising model. What is the maximum size L that would be possible to study in 24 hours on a single 3GHz-processor. Base your answer on some reasonable assumptions. (2p)

5 Scaling analysis

One way to analyze experimental data (or simulations at big lattices) is to plot $m/|t|^a$ versus $h/|t|^c$. Start from $m \sim \partial f/\partial h$ and (4p)

$$f(t,h) = b^{-d} f(tb^{y_t}, hb^{y_h}),$$

and express a and c in terms of d, y_t, and y_h.

(The formulas below are perhaps not really needed.)

	definition	With d, y_t, and y_h
α	$C \sim \|T - T_c\|^{-\alpha}$	$2 - d/y_t$
β	$m \sim (T_c - T)^{\beta}$	$(d - y_h)/y_t$
γ	$\chi \sim \|T - T_c\|^{-\gamma}$	$(2y_h - d)/y_t$
δ	$m \sim h^{1/\delta}, \quad T = T_c$	$y_h/(d - y_h)$
η	$g(k) \sim 1/k^{2-\eta}, \quad T = T_c$	$d + 2 - 2y_h$
ν	$\xi \sim \|T - T_c\|^{-\nu}$	$1/y_t$