4.12 Superconductivity and the XY model

Overview:

- Short historical note.
- Ginzburg-Landau theory.
- The XY model and superconductivity.

4.12 Superconductivity and the XY model

- Discovered in 1911 by H. Kammerlingh-Onnes through measurements on mercury at very low temperatures.
- The electrical resistivity appeared to vanish below 4.2 K.
- Superconductivity has since then been been found in many different materials, both elements and alloys.
- The resitivity is actually identically zero.
- Hallmark of a superconductor: *perfect* conductivity.
- Another equally important phenomenon: the Meissner effect—the magnetic field is expelled from the superconductor.

Theories

- The microscopic mechanism of superconductivity was found 1957.
- Bardeen, Cooper, and Schrieffer—the BCS-theory.
    - The electrons are bound together in pairs—Cooper pairs—by a weak force caused by phonons.
    - These paired electrons are the superconducting charge carriers.
- Nobel prize in 1972.
- The phenomenological Ginzburg-Landau theory of superconductivity—much used in spite of the existence of a microscopic theory.

Ginzburg-Landau theory of superconductivity

Two assumptions:

- A superconductor at each point is characterized by a complex "order parameter" $\psi(\mathbf{r})$,
- The free energy may be written in powers of ψ and $\nabla\psi$,

$$F = \int \mathbf{dr} \left[\frac{\mathbf{B}^2}{8\pi} + \frac{\hbar^2}{2m^*} \left| \left(\nabla - \frac{ie^*}{\hbar c} \mathbf{A} \right) \psi \right|^2 + \alpha(T)|\psi|^2 + \frac{\beta}{2}|\psi|^4 \right],$$

  - $m^* = 2m$ and $e^* = 2e$ are the mass and charge of a Cooper pair,
  - $\mathbf{B} = \nabla \times \mathbf{A}$ is the magnetic induction,
  - the function $\alpha(T)$ goes negative at T_{c0}. Often approximated $\alpha(T) = \alpha'(T - T_{c0})$.

Ginzburg-Landau theory of superconductivity. . . cont'd

- The equilibrium value of $|\psi|$ is determined by the condition that the free energy is a minimum. In the absence of external fields this becomes
$$|\psi|^2 = -m^* \frac{\alpha(T)}{\beta}.$$
- The expression for the current becomes similar to the expression from common quantum mechanics:

$$\mathbf{j} = -\frac{ie^*\hbar}{2m^*}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{(e^*)^2}{m^*c}|\psi|^2\mathbf{A} = \frac{e^*}{m^*}|\psi|^2\left(\hbar\nabla\theta - \frac{e^*}{c}\mathbf{A}\right),$$

For us the take-away message is

$$\mathbf{j} \sim \nabla\theta.$$

The XY model and superconductivity

A few steps lead to the XY model:

- Neglect the magnetic energies.
- Assume that the spatial variation in $|\psi|$ is unimportant. This means that $\theta(\mathbf{r})$ is the only remaining degree of freedom.
- Discretize space such that $\theta(\mathbf{r})$ is only defined at certain lattice points $\mathbf{r}_i$. Use the notation θ_i.
- Change the interaction potential, (with i and j nearest neighbors),

$$|\nabla\psi(\mathbf{r})|^2 \longrightarrow -J\cos(\theta_i - \theta_j),$$

The resulting Hamiltonian is

$$H_{XY} = -J\sum_{\langle ij\rangle}\cos(\theta_i - \theta_j).$$

For small $\Delta\theta \equiv \theta_i - \theta_j$: $-\cos\Delta\theta \approx -[1-(\Delta\theta)^2/2] \sim \frac{(\Delta\theta)^2}{2}$.

Properties of the 2D XY model

Spin waves: The spin directions of a configuration may change slowly and gradually as one moves in space— spin waves.

The effect can be calculated analytically:

$$g(\mathbf{r}) = e^{TG(\mathbf{r})/J} \sim r^{-T/2\pi J},$$

Here $G(\mathbf{r})$ is "the lattice Green's function in two dimensions".
The final result follows from $2\pi G(\mathbf{r}) \approx -\ln|\mathbf{r}|$.

Conclusion: the 2D XY model has no long range order and therefore no non-trivial phase transition.

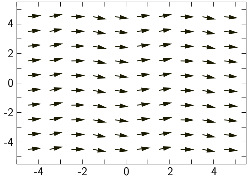
Properties of the 2D XY model…cont'd

Quasi long range order:
The model nevertheless has a phase transition!
Nobel prize to Kosterlitz and Thouless 2016.
The vortex unbinding transition:

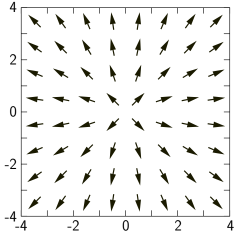
- Low temperature: quasi long range order.
- Above the transition: there start to appear single vortices.

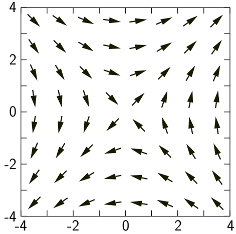
Properties of the 2D XY model…cont'd

Spin waves and vortices



Vortices come in two kinds that are usually close togeter.





Interesting phenomenon in thin superconducting films

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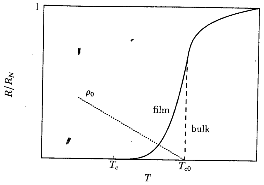


Figure 4: Sketch of the resistivity as a function of temperature for both thin film (solid line) and bulk (dashed line). The dotted line shows the temperature dependence of ρ_0, the density of superconducting electrons.

In thin films there is resistance even though there are lots of superconductiong electons!

How can that be?

An effect of the presence of free vortices.

- Consider making a thin superconducting film into a cylinder.
- Set up a state with a current, i.e. a finite $\nabla\theta$.
- The phase will wind an integer number around the cylinder.

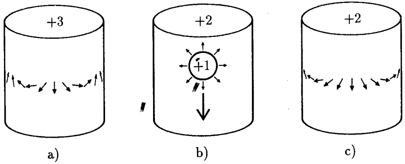


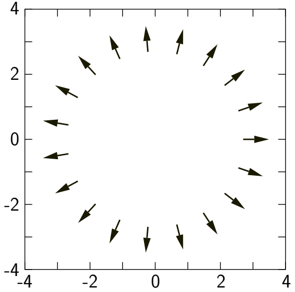
Figure 5: The winding number may change if a vortex – an object with winding number ±1 – forms at the boundary, moves across the sample, and crosses the second boundary. In a superconducting films this process leads to a degradation of the current, which is perceived as a resistivity in the sample.

The current may be reduced only if an object with unity winding number—a vortex—moves across the system.

Energy-entropy balance for the 2D XY model

What is the energy cost for a vortex in an $L \times L$ system?

- Put the spins on concentric circles at distance 0.5, 1.5, 2.5,. . . .
- For circle at distance R: perimeter $p = 2\pi R$.
- p neighbor pairs with angular difference $= \Delta\theta = 2\pi/p = 1/R$.
- Energy at distance R: $E(R) = pJ(\Delta\theta)^2/2 = \pi J/R$.



Energy-entropy balance for the 2D XY model...cont'd

Total energy for a vortex:

$$E_{\text{vortex}} = \int_{1/2}^{L/2} E(R)dR = J\int_{1/2}^{L/2} \frac{\pi}{R}dR = J\pi\left(\ln\frac{L}{2} - \ln\frac{1}{2}\right) = \pi J \ln L.$$

Number of places for a vortex $= L^2$ gives $S = \ln L^2$.

Single (unbound) vortices will start to exist when the free energy for a vortex turn negative:

$F = E_{\text{vortex}} - TS = \pi J \ln L - 2T\ln L$ becomes negative when $T > \pi J/2$.

The Kosterlitz-Thouless temperature: $T_{\text{KT}}/J = \pi/2$.