7.1.6 Circuit for van der Pol oscillator

Consider a circut with three parts (and current direction 1–2–3–1).

1-2 Inductor, L, governed by $V_{12} = L\dot{I}$.

2-3 Capacitor, C, with $V_{23} = Q/C$.

3-1 Resistor, with $V_{31} = f(I)$.

Suppose a source of current is attached to the circuit and then withdrawn. What equations govern the subsequent evolution of the current and the various voltages?

(a) Let $V = V_{32}$ denote the voltage drop from point 3 to point 2 in the circuit. Show that $\dot{V} = -I/C$ and $V = L\dot{I} + f(I)$.

We have

$$\dot{V} = -\dot{Q}/C = -I/C, \tag{1}$$

and, with $V \equiv V_{32} = -V_{23} = -V_{21} - V_{13} = V_{12} + V_{31}$:

$$V = L\dot{I} + f(I), \quad \Rightarrow \dot{I} = \frac{V}{L} - \frac{f(I)}{L}. \tag{2}$$

(b) Show that the equations in (a) are equivalent to

$$\frac{dw}{d\tau} = -x, \quad \frac{dx}{d\tau} = w - \mu F(x).$$

where $x = L^{1/2}I$, $w = C^{1/2}V$, $\tau = (LC)^{-1/2}$, and $F(x) = f(L^{-1/2}x)$.

Plug in

$$I = x/L^{1/2}, \quad V = w/C^{1/2}, \quad dt = (LC)^{1/2}d\tau,$$

into the equations above:

$$(1) \quad \Rightarrow \quad \frac{dw/C^{1/2}}{d\tau(LC)^{1/2}} = -\frac{1}{C}\frac{x}{L^{1/2}} \Rightarrow \frac{dw}{d\tau} - x,$$

$$(2) \quad \Rightarrow \quad \frac{dx/L^{1/2}}{d\tau(LC)^{1/2}} = \frac{w/C^{1/2}}{L} - \frac{1}{L}f(x/L^{1/2}) \Rightarrow \frac{dx}{d\tau} = w - C^{1/2}F(x).$$

This is claimed to be equivalent to the van der Pol equation if $F(x) = x^3/3 - x$.

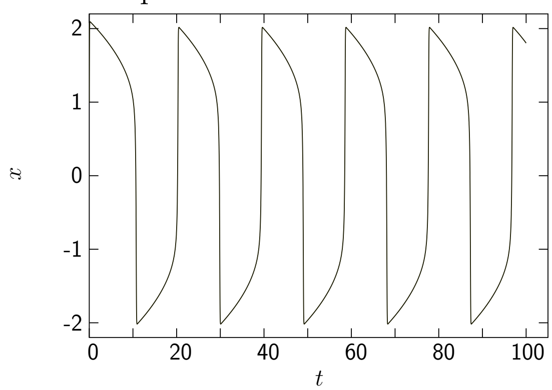
The van der Pol oscillator

This is from Example 7.5.1 in Strogatz.

Introduce $\mu = C^{1/2}$ and take $y = w/\mu$. We also demand $\mu \gg 1$. Then

$$\begin{aligned} \dot{x} &= \mu(y - F(x)), \\ \dot{y} &= -x/\mu. \end{aligned}$$

The time dependence becomes



(In Stogatz they determine the general behavior by analyzing the nullclines.)