6.5.6 Epidemic model

In this problem we will reanalyze Exerc. 3.7.6 in the phase plane. We let $x(t)$ denote the size of the healthy population and $y(t)$ the size of the sick population. The model, with parameters $k, \ell > 0$, is

$$\begin{cases} \dot{x} &= -kxy, \\ \dot{y} &= kxy - \ell y. \end{cases}$$

a) *Find and classify all fixed points!*
Fixed points: $(x, y) = (c, 0)$ with constant $c > 0$. For general x, y the matrix is

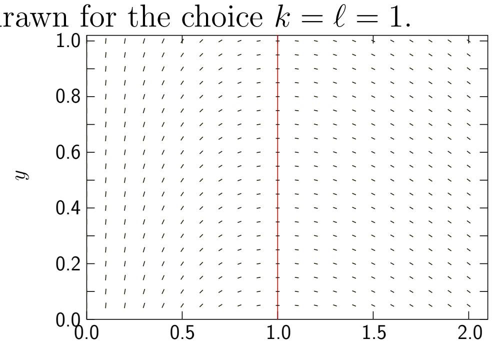
$$\begin{pmatrix} -ky & -kx \\ ky & kx - \ell \end{pmatrix}.$$

which, when evaluated at $(c, 0)$, becomes

$$\begin{pmatrix} 0 & -kc \\ 0 & kc - \ell \end{pmatrix}.$$

and gives $\tau = kc - \ell$ and $\Delta = 0$. the eigenvalues become $\lambda_1 = kc - \ell$ and $\lambda_2 = 0$. A line of fixed points!

b) *Sketch the nullclines and the vector field!*
(Arrows are missing, but the flow is always to smaller x.) The figure is drawn for the choice $k = \ell = 1$.



c) *Find a conserved quantity for the system! (Hint: Form a differential equation for dy/dx. Separate the variables and integrate both sides.)*
Conserved quantity:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{kxy - \ell y}{-kxy} = \frac{\ell}{kx} - 1.$$

Rearrange:

$$dy = \frac{\ell}{k}\frac{dx}{x} - dx,$$

which integrates to

$$y = \frac{\ell}{k}\ln x - x + C.$$

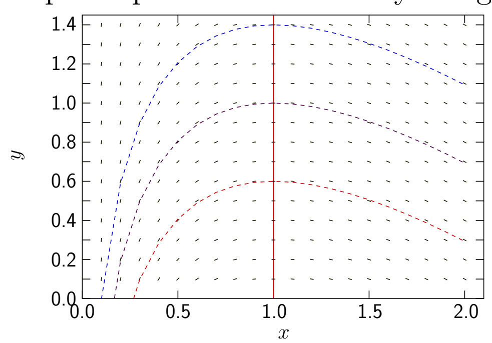
This means that the conserved quantity is

$$y + x - \frac{\ell}{k}\ln x,$$

but also that we get an explicity expression for y:

$$y = C + \frac{\ell}{k}\ln x - x.$$

d) The phase portrait obtained by using $C = 1.6$, 2.0, and 2.4.



As $t \to \infty$ the infection goes away with a certain remaining population.

e) An epidemic occurs if $\dot{y} > 0$, initially, which is the case if $x(0) > \ell/k$.