6.5.1 Conserved quantity

Consider the equation

$$\ddot{x} = x^3 - x.$$

When rewritten as two first order equations it becomes

$$\begin{cases} \dot{x} &= v, \\ \dot{v} &= x^3 - x. \end{cases}$$

The Jacobian matrix becomes

$$\begin{pmatrix} 0 & 1 \\ 3x^2 - 1 & 0 \end{pmatrix}.$$

a) *Find all the equilibrium points and classify them.*
Fixed points:

i) $(x, v) = (0, 0)$:

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Rightarrow \quad \tau = 0, \quad \Delta = 1 \quad \Rightarrow \quad \lambda_\pm = \pm i.$$

Which makes us conclude that this fixed point is a center.

ii) $(x, v) = (1, 0)$:

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \quad \Rightarrow \quad \tau = 0, \quad \Delta = -2 \quad \Rightarrow \quad \lambda_\pm = \pm\sqrt{2},$$

and we conclude that this is a saddle point.
We then determine the eigenvectors $u_\pm$: From $Au_+ = \lambda_+ u_+$:

$$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \sqrt{2} \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \quad \begin{pmatrix} b \\ 2a \end{pmatrix} = \begin{pmatrix} \sqrt{2}a \\ \sqrt{2}b \end{pmatrix} \Rightarrow \quad u_+ = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$

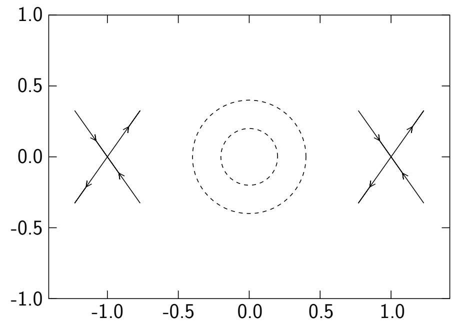
Similar algebra gives

$$u_- = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}.$$

Note that u_+ is the repulsive direction whereas u_- is the attractive direction.

iii) $(x, v) = (-1, 0)$: The matrix is the same as for $(1, 0)$ and the behavior is therefore the same.

The behavior close to these fixed points are shown below:



b) *Find a conserved quantity.*
To find a conserved quantity we note that $\dot{v} = x^3 - x$ implies that $v\dot{v} + \dot{x}x - \dot{x}x^3 = 0$, which can be rewritten as

$$\frac{d}{dt}\left[\frac{v^2}{2} + \frac{x^2}{2} - \frac{x^4}{4}\right] = 0,$$

which shows that the conserved quantity is

$$\frac{v^2}{2} + \frac{x^2}{2} - \frac{x^4}{4}.$$

For small x, such that x^4 may be neglected, this is the equation for a circle in the (x, v) plane. And this makes sense since the fixed point at the origin is a center.

c) *Sketch the phase portrait.*
The figure below shows some trajectories obtained by integrating the equations but they could of course equally well have been obtained from Eq. (1) above. Each curve corresponds to a certain C-value.