7. Limit cycles—introduction

There are many systems that display self-sustained oscillations:

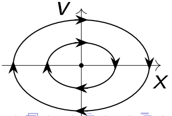
- the beating of the heart,
- daily rythms in the human body temperature and hormone secretion,
- dangerous vibrations in bridges and airplane wings,
- "hunting oscillations" in railway wheels.

Standard oscillations of a certain period and amplitude!

A limit cycle—an isolated closed trajectory.

- "Isolated" — neighboring trajectories are not closed, they spiral toward or away from the limit cycle.
- Inherently nonlinear phenomena! In linear systems closed orbits will never be isolated. If $\mathbf{x}(t)$ is a solution then is $c\mathbf{x}(t)$.

The harmonic oscillator (illustrated) is not an example of a limit cycle.
The amplitude of the oscillations will depend on the initial conditions.

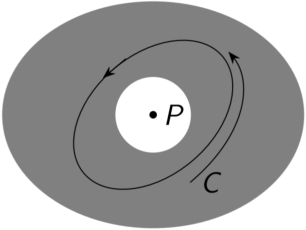


7.3 Poincaré-Benedixson theorem

Note: The following is relevant for to two dimensions—the phase plane.

Suppose that

1. R is a closed, bounded subset of the plane,
2. $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is a continuously differentiable vector field on an open set containing R,
3. R does not contain any fixed points,
4. There exists a trajectory C that is confined in R—it starts in R and stays in R for all future times.



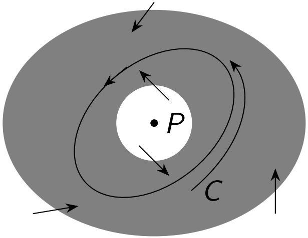
Then either C is a closed orbit, or it spirals toward a closed orbit as $t \to \infty$.

In any case R **contains a closed orbit.**

7.3 Poincaré-Benedixson theorem. . . cont'd

Note:

- Any closed orbit must encircle a fixed point, therefore R is a ring-shaped region.
- It is easy to satisfy points 1—3.
- Point 4 is the tough one:
  Construct a trapping region such that the vector field points "inward" everywhere on the boundary of R. The trajectory is then confined in R.



This can be used to prove that there is a closed orbit in a certain region.

7.3 Poincaré-Benedixson theorem. . . and the impossibility of chaos

Two dimensions is special (different from higher dimensions): a simple closed curve subdivides a plane into two disjoint open regions.

When the Poincaré-Benedixson theorem applies the trajectory must eventually approach a closed orbit.

$\Rightarrow$ nothing complicated is possible. . .
and we conclude that **chaos is not possible** in two dimensions.

(Chaos would imply complicated trajectories that depend sensitively on the starting point.)

When should we expect a limit cycle?

If there is a closed orbit there is also a fixed point, inside.

If this fixed point is unstable we should expect a limit cycle.

Are these two conditions sufficient to *guarantee* the existence of a limit cycle? Presumably not, but that is anyway what one would expect.

There could perhaps be other possibilities as e.g. a set of closed orbits. (The textbook is not clear on this.)