7. Limit cycles—introduction

There are many systems that display self-sustained oscillations:

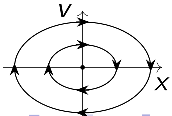
- the beating of the heart,
- daily rythms in the human body temperature and hormone secretion,
- dangerous vibrations in bridges and airplane wings,
- "hunting oscillations" in railway wheels.

Standard oscillations of a certain period and amplitude!

A limit cycle—an isolated closed trajectory.

- "Isolated" — neighboring trajectories are not closed, they spiral toward or away from the limit cycle.
- Inherently nonlinear phenomena! In linear systems closed orbits will never be isolated. If $\mathbf{x}(t)$ is a solution then is $c\mathbf{x}(t)$.

The harmonic oscillator (illustrated) is not an example of a limit cycle.
The amplitude of the oscillations will depend on the initial conditions.

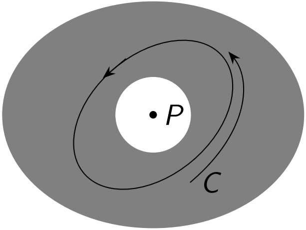


7.3 Poincaré-Benedixson theorem

Note: The following is relevant for to two dimensions—the phase plane.

Suppose that

1. R is a closed, bounded subset of the plane,
2. $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is a continuously differentiable vector field on an open set containing R,
3. R does not contain any fixed points,
4. There exists a trajectory C that is confined in R—it starts in R and stays in R for all future times.



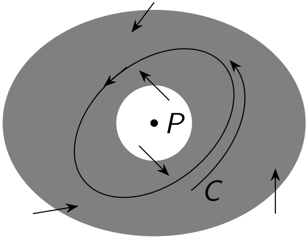
Then either C is a closed orbit, or it spirals toward a closed orbit as $t \to \infty$.

In any case R **contains a closed orbit.**

7.3 Poincaré-Benedixson theorem. . . cont'd

Note:

- Any closed orbit must encircle a fixed point, therefore R is a ring-shaped region.
- It is easy to satisfy points 1—3.
- Point 4 is the tough one:
  Construct a trapping region such that the vector field points "inward" everywhere on the boundary of R. The trajectory is then confined in R.



This can be used to prove that there is a closed orbit in a certain region.

7.3 Poincaré-Benedixson theorem. . . and the impossibility of chaos

Two dimensions is special (different from higher dimensions): a simple closed curve subdivides a plane into two disjoint open regions.

When the Poincaré-Benedixson theorem applies the trajectory must eventually approach a closed orbit.

$\Rightarrow$ nothing complicated is possible. . .
and we conclude that **chaos is not possible** in two dimensions.

(Chaos would imply complicated trajectories that depend sensitively on the starting point.)

When should we expect a limit cycle?

If there is a closed orbit there is also a fixed point, inside.

If this fixed point is unstable we should expect a limit cycle.

Are these two conditions sufficient to *guarantee* the existence of a limit cycle? Presumably not, but that is anyway what one would expect.

There could perhaps be other possibilities as e.g. a set of closed orbits. (The textbook is not clear on this.)

The Lotka-Volterra predator prey model

The assumptions of the model are:

- In the absence of foxes the rabbits breed with $dR/dt = aR$.
- The foxes reduce the growth rate by a term bFR.
- The relative growth rate of the fox population is proportional to the population of rabbits, $dF/dt = cFR$.
- The foxes have a certain death rate, dF. (Here "d" is a constant.)

Taken together this becomes $\begin{cases} \frac{dR}{dt} = aR - bFR, \\ \frac{dF}{dt} = cFR - dF. \end{cases}$

This is the Lotka-Volterra system of equations.

- Used by Volterra in 1926 to explain the oscillatory levels of certain fish catches.
- Derived by Lotka to describe a hypotetical chemical reaction in 1920.

The LV model…Rewrite in dimensionless units

To simplify the analysis we rewrite the equations with

$$f = \frac{bF}{a}, \quad r = \frac{cR}{d}, \quad \tau = at, \quad \alpha = \frac{d}{a},$$

and they become

$$\frac{dr}{d\tau} = r(1-f), \tag{1}$$

$$\frac{df}{d\tau} = \alpha f(r-1). \tag{2}$$

This gives two fixed points:

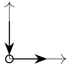
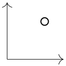
$$(r^*, f^*) = (0,0), \quad (r^*, f^*) = (1,1).$$

The LV model…Fixed points and their properties

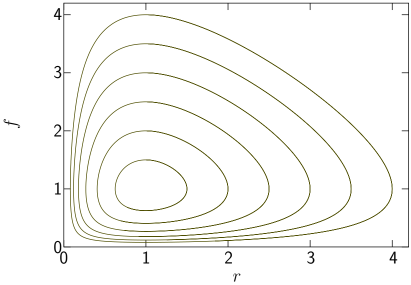
For the predator prey model we find the Jacobian matrix

$$\mathbf{A} = \begin{pmatrix} 1-f & -r \\ \alpha f & \alpha(r-1) \end{pmatrix},$$

and for the two fixed points:

- $(0,0)$: $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -\alpha \end{pmatrix} \Rightarrow \begin{matrix} \tau = 1-\alpha, \\ \Delta = -\alpha, \end{matrix} \Rightarrow$ saddle point: 
- $(1,1)$: $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ \alpha & 0 \end{pmatrix} \Rightarrow \begin{matrix} \tau = 0, \\ \Delta = \alpha, \end{matrix} \Rightarrow$ center: 

The LV model. . . Not very realistic!



This simple Lotka-Volterra model should not be taken too seriously!

- Real predator-prey cycles typically have a characteristic amplitude.
- Therefore, realistic models should predict a *single* closed orbit, but not a continuous family of neutrally stable cycles.

Limit cycle in a more realistic predator prey model

Two unrealistic assumptions in the Lotka-Volterra model:

- growth of rabbits is unbounded in the absence of predation,
- there is no limit to the prey consumption.

Define modified equations with two more parameters, r_{grass} (the carrying capacity) and r_{sat}:

$$
\begin{aligned}
\frac{dr}{d\tau} &= f_1(r,f) = r(1 - r/r_{\text{grass}}) - \frac{rf}{1 + r/r_{\text{sat}}}, && \text{compare } r - rf. \\
\frac{df}{d\tau} &= f_2(r,f) = \alpha\frac{rf}{1 + r/r_{\text{sat}}} - \alpha f, && \text{compare } \alpha rf - \alpha f.
\end{aligned}
$$

With properly chosen r_{grass} and r_{sat} it is possible to find a stable limit cycle.