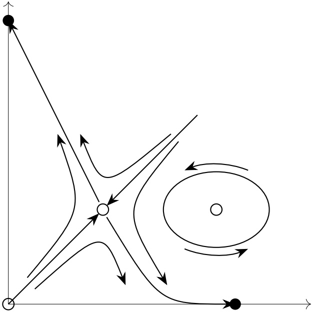
6. Phase plane

General two dimensional. Compact vector notation, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$.

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2), \\ \dot{x}_2 &= f_2(x_1, x_2). \end{aligned}$$

Many possibilities:



Things that characterize the system:

- Arrangements of fixed points; $\dot{x} = 0$.
- Stable, unstable, or saddle point?
- Closed orbits—periodic solutions, $\mathbf{x}(t + T) = \mathbf{x}(t)$.

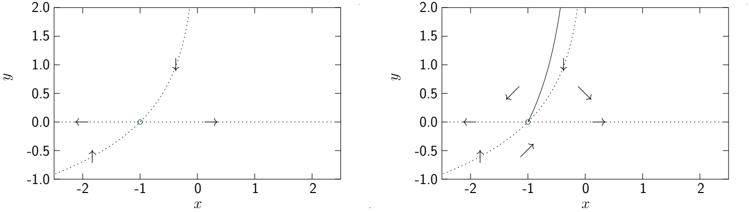
6.1 Graphical analysis

To get a quick understanding of the behavior it is clever to first sketch the two “nullclines”, i.e. the lines along which $\dot{x} = 0$ or $\dot{y} = 0$. The crossing of these lines give the fixed points.

For the equations

$$\begin{aligned} \dot{x} &= x + e^{-y}, \\ \dot{y} &= -y, \end{aligned} \quad \text{we get two nullclines:} \quad \begin{aligned} x &= -e^{-y}, \\ y &= 0. \end{aligned}$$

Plot the nullclines and the flow directions:



Note: nullclines are usually not the same as the trajectories.

6.2 Existence and uniqueness of solutions

Theorem:

Consider the initial value problem

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(0) = \mathbf{x}_0.$$

Suppose that $\mathbf{f}(\mathbf{x})$ *and all its partial derivatives are continuous in some open connected set. Then for* $\mathbf{x}$ *on this set the initial value problem has a solution* $\mathbf{x}(t)$ *on some time interval* $(-\tau, \tau)$ *about* $t = 0$ *and the solution is unique.*

Important corollary:

different trajectories never intersect. (If they did, the uniqueness part of the theorem would be violated.)

In two dimensions: If there is a closed orbit, any trajectory starting inside must remain there forever. Such trajectories will either approach a fixed point (inside the closed orbit) or approach the closed orbit.

6.3 Fixed points and linearization

Consider

$$\begin{aligned}\dot{x} &= f(x,y),\\ \dot{y} &= g(x,y).\end{aligned}$$

Suppose (x^*, y^*) is a fixed point, $f(x^*, y^*) = 0$ and $g(x^*, y^*) = 0$.

Notation for small deviations: $u = x - x^*$, $v = y - y^*$:

$$\dot{u} = f(x^* + u, y^* + v) = f(x^*, y^*) + u \left.\frac{\partial f}{\partial x}\right|^* + v \left.\frac{\partial f}{\partial y}\right|^* + O(u^2, v^2, uv).$$

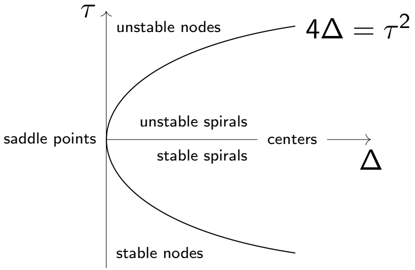
$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \text{quadratic terms.}$$

The *Jacobian matrix* of the fixed point (x^*, y^*)—similar to $f'(x)$ in 1D:

$$\mathbf{A} = \begin{pmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{pmatrix}.$$

The nonlinear terms? May be neglected if not on the borderline cases.

Last lecture: 5.2 General classification



- If $\Delta < 0$ the eivenvalues are real with opposite signs: saddle point.
- if $\Delta > 0$: stable if $\tau < 0$ (node or spiral)
  - when $4\Delta < \tau^2$: eigenvalues real with the same sign (nodes)
  - when $4\Delta > \tau^2$: complex conjugate (spirals and centers)

Example 6.3.1

Find and classify the fixed points of the system $\dot{x} = -x + x^3$, $\dot{y} = -2y$.
From $x = x^3$ and $y = 0 \Rightarrow$ Nullclines: $x = -1, 0, 1$ and $y = 0$.
They cross at the three fixed points: $(-1, 0)$, $(0, 0)$, and $(1, 0)$.

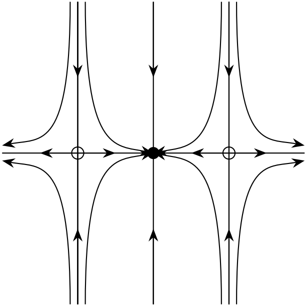
The Jacobian matrix: $\mathbf{A} = \begin{pmatrix} -1 + 3x^2 & 0 \\ 0 & -2 \end{pmatrix}$.

Evaluate for the three fixed points:

- $(-1, 0)$: $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$, $\tau = 0$, $\Delta = -4 \Rightarrow$ saddle point.
- $(0, 0)$: $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$, $\tau = -3$, $\Delta = 2 \Rightarrow$ stable node.
- $(1, 0)$: $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$, $\tau = 0$, $\Delta = -4 \Rightarrow$ saddle point.

Example 6.3.1. . . cont'd

Approximate trajectories:



Nullclines: $x = -1$, 0, 1 and $y = 0$.

Fixed points:

- $(-1, 0)$, saddle point,
- $(0, 0)$, stable node,
- $(1, 0)$, saddle point.

6.4 Rabbits versus sheep

Suppose that both species are competing for the same food.

- Each species would grow to its carrying capacity in the absence of the other.
- When they encounter each other, trouble starts. Usually the sheep nudges the rabbit aside.
  Assume that these conflicts
  - occur at a rate proportional to the size of each populations,
  - affects the growth rate more severaly for rabbits than for sheep.

A model that captures these assumptions:

$$\begin{cases} \dot{x} = x(3 - x - 2y), \quad \text{rabbits}, & [\dot{x} = 3x(1 - x/3)], \\ \dot{y} = y(2 - x - y), \quad \text{sheep}, & [\dot{y} = 2y(1 - y/2)]. \end{cases}$$

Four fixed points: $(x, y) = (0, 0)$, $(0, 2)$, $(3, 0)$, and $(1, 1)$.

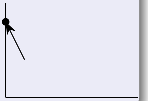
The Jacobian becomes $\quad \mathbf{A} = \begin{pmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - x - 2y \end{pmatrix}.$

1. $(x^*, y^*) = (0,0)$: unstable node, $\lambda_1 = 3$, $\lambda_2 = 2$.

$$\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, \quad \begin{matrix} \tau = 5, \\ \Delta = 6, \end{matrix} \quad \mathbf{v}_1 = (0,1),\ \mathbf{v}_2 = (1,0).$$

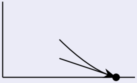
2. $(x^*, y^*) = (0,2)$, stable node, $\lambda = -1, -2$.

$$\mathbf{A} = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}, \quad \begin{matrix} \tau = -3, \\ \Delta = 2, \end{matrix} \quad \mathbf{v}_1 = (1,-2),\ \mathbf{v}_2 = (0,1).$$



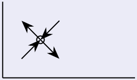
3. $(x^*, y^*) = (3,0)$ stable node, $\lambda = -3, -1$.

$$\mathbf{A} = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}, \quad \begin{matrix} \tau = -4, \\ \Delta = 3, \end{matrix} \quad \mathbf{v}_1 = (1,0),\ \mathbf{v}_2 = (3,-1).$$

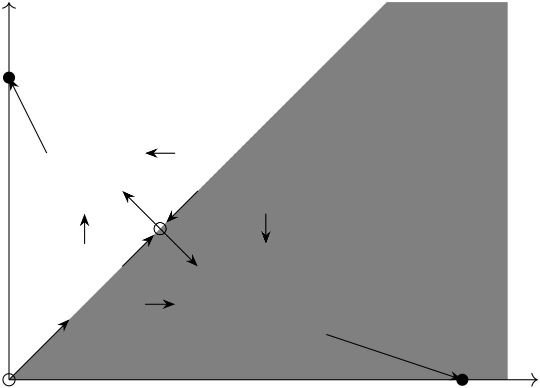


4. $(x^*, y^*) = (1,1)$, saddle point, $\lambda = 1 \pm \sqrt{2}$

$$\mathbf{A} = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}, \quad \begin{matrix} \tau = -2, \\ \Delta = -1 \end{matrix}, \quad \begin{matrix} \mathbf{v}_1 = (-1, 1 + \sqrt{2}/2) \\ \mathbf{v}_2 = (-1, 1 - \sqrt{2}/2) \end{matrix}$$



Combine all together. Easy to add some more flow directions.



The gray area—the "basin of attraction" for the fixed point at $(3, 0)$.
Defined to be the set of initial conditions $\mathbf{x}_0$ such that $\mathbf{x}(t \to \infty) \to \mathbf{x}^*$.

6.5 Conservative systems

Consider Newton's law,

$$m\ddot{x} = F(x), \quad \text{where } F \text{ is independent of both } \dot{x} \text{ and } t.$$

We can then show that energy is conserved by introducing the *potential energy* $V(x)$, defined by

$$F(x) = -dV/dx, \quad \text{which gives} \quad m\ddot{x} + \frac{dV}{dx} = 0.$$

Multiply by $\dot{x}$ on both sides!

$$m\dot{x}\ddot{x} + \frac{dV}{dx}\dot{x} = 0 \quad \Rightarrow \quad \frac{d}{dt}\left[\frac{1}{2}m\dot{x}^2 + V(x)\right] = 0,$$

which follows since

$$\frac{d}{dt}V(x(t)) = \frac{dV}{dx}\frac{dx}{dt}.$$

6.5 Conservative systems. . . cont'd

For a given solution $x(t)$ the quantity $\frac{1}{2}m\dot{x}^2 + V(x)$ is a constant of time. We identify this with the energy,

$$E = \frac{1}{2}m\dot{x}^2 + V(x).$$

More precise and general:

> *Given a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, a conserved quantity is a real-valued continuous function $E(\mathbf{x})$ that is constant on trajectories, i.e. $dE/dt = 0$.*
>
> *We also require that $E(\mathbf{x})$ is nonconstant on all open sets.*

6.5 Conservative systems. . . cont'd

Conservative systems cannot have any attracting fixed point.

Show this:

Suppose that $\mathbf{x}^*$ *were an attracting fixed point. Then*

- *all points in its basin of attraction would have the same energy,*
- *and therefore* $E(\mathbf{x})$ *is a constant function for* $\mathbf{x}$ *in this basin.*

This contradicts the requrement that $E(\mathbf{x})$ *is nonconstant on all open sets.*

One instead expects to find saddles and centers.