II Two-dimensional flows

Start with linear systems in Ch. 5 introduce nonlinearities in Ch. 6.

Two kinds of 2D problems:

- variables are different things, rabbits, foxes,
- cases where n = 2 comes from the second order equation, $\ddot{x} = \dots$

5 Linear systems

General twodimensional:

$$\dot{x}_1 = f_1(x_1, x_2),$$

 $\dot{x}_2 = f_2(x_1, x_2).$

Specialize to linear systems (with $x_1 \rightarrow x$ and $x_2 \rightarrow y$):

$$\dot{x} = ax + by,$$

 $\dot{y} = cx + dy.$

Matrix notation:

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$
$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

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Graphical analysis in 2D—(a) harmonic oscillator

Consider $m\ddot{x} = -kx$:

$$\begin{cases} \dot{x} = v, \\ \dot{v} = -\omega^2 x, \quad \omega^2 = k/m. \end{cases}$$

k m

which implies

 $(\dot{x}, \dot{v}) = (v, -\omega^2 x)$, a vector to each point (x, v).

To find a trajectory, put an imaginary particle at (x_0, v_0) and see how it is carried around by the flow. "Phase portrait"



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Graphical analysis in 2D—(b) uncoupled equations

Consider

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$
$$\begin{cases} \dot{x} = ax \\ \dot{y} = -y \end{cases}$$

Solved separaterly:

$$\begin{aligned} x(t) &= x_0 e^{at}, \quad \stackrel{a > 0}{\longleftrightarrow} \quad \text{or} \quad \stackrel{a < 0}{\longrightarrow} \\ y(t) &= y_0 e^{-t}, \quad \uparrow. \end{aligned}$$

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Image: A matrix

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Uncoupled equations...many different possible behaviors



Uncoupled equations...many different possible behaviors



Stability language:

- (i) globally attracting as in a-c: all that starts near \mathbf{x}^* approach it as $t \to \infty$.
- (ii) Liapunov stable starting close to \bar{x}^* remains close for all times.

More terms:

- neutrally stable: (ii) but not (i). Examples: (d) and (0,0) in the harmonic oscillator.
- stable: both (i) and (ii).
- unstable: neither (i) nor (ii).

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5.2 Classification of linear systems

In the uncoupled case the x- and y- axes had straight trajectories. Starting on one of them we stayed on it forever.

Now with
$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
, look for trajectories $\mathbf{x}(t) = e^{\lambda t} \mathbf{v}$.
Put into $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$:

$$\lambda e^{\lambda t} \mathbf{v} = e^{\lambda t} \mathbf{A} \mathbf{v} \quad \Rightarrow \quad \lambda \mathbf{v} = \mathbf{A} \mathbf{v} \quad - \text{eigenvalue equation}$$

To determine the eigenvectors:

$$\begin{bmatrix} \mathbf{A} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{bmatrix} \mathbf{v} = \mathbf{0},$$

which is solved through the charcteristic equation,

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0.$$

$$\det\begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = 0 \quad \Rightarrow \quad (a - \lambda)(d - \lambda) - bc = 0.$$

$$\lambda^{2} - \tau\lambda + \Delta = 0, \quad \text{where } \tau = \text{Tr}\mathbf{A} = a + d, \quad \Delta = \det\mathbf{A} = ad - bc.$$
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5.2 Classification of linear systems... cont'd

With $\tau = \text{Tr} \mathbf{A} = a + d$ and $\Delta = \det \mathbf{A} = ad - bc$,

$$(a-\lambda)(d-\lambda)-bc=0 \quad \Rightarrow \quad \lambda^2-\lambda(a+d)+(ad-bc)=0,$$

this may be rewritten

$$\lambda^2 - \tau \lambda + \Delta = \mathbf{0},$$

with solutions

$$\lambda_1 = \frac{\tau}{2} + \frac{1}{2}\sqrt{\tau^2 - 4\Delta}, \quad \lambda_2 = \frac{\tau}{2} - \frac{1}{2}\sqrt{\tau^2 - 4\Delta}.$$

Two eigenvectors: $\mathbf{A}\mathbf{v}_1 = \lambda_1\mathbf{v}_1$, $\mathbf{A}\mathbf{v}_2 = \lambda_2\mathbf{v}_2$.

The eigenvalues are typically distinct, $\lambda_1 \neq \lambda_2$. The eigenvectors are then independent and span the plane. Any initial condition can then be written

$$\mathbf{x}_0 = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2, \quad \Rightarrow \quad \mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2.$$

5.2 Classification of linear systems. . . examples Ex. 1 (5.2.1 and 5.2.2): Solve

$$\begin{cases} \dot{x} = x + y, \\ \dot{y} = 4x - 2y. \end{cases}$$

 $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \Rightarrow \tau = -1, \quad \Delta = -6 \quad \text{which gives } \lambda_1 = 2, \quad \lambda_2 = -3.$

Find eigenvectors, $\mathbf{v} = \begin{pmatrix} u \\ w \end{pmatrix}$:

$$\left(\begin{array}{c}0\\0\end{array}\right)=\left(\begin{array}{c}1-\lambda&1\\4&-2-\lambda\end{array}\right)\left(\begin{array}{c}u\\w\end{array}\right)=\left(\begin{array}{c}(1-\lambda)u+w\\4u-(2+\lambda)w\end{array}\right).$$

This gives $w = (\lambda - 1)u$ which can be used to determine the (unnormalized) eigenvectors:

$$\lambda_1 = 2 \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda_2 = -3 \Rightarrow \mathbf{v}_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

With $\lambda_1 = 2$ and $\lambda_2 = -3$ together with

$$\mathbf{v}_1=\left(egin{array}{c}1\\1\end{array}
ight)$$
 and $\mathbf{v}_2=\left(egin{array}{c}1\\-4\end{array}
ight)$



This is a saddle point! The *stable manifold* is the set of points along $\pm \mathbf{v}_2$. The *unstable manifold* is the set of points along $\pm \mathbf{v}_1$.

5.2 Classification of linear systems: Example 2

$$\lambda_{1,2} = rac{ au}{2} \pm rac{1}{2} \sqrt{ au^2 - 4\Delta}, \quad \Rightarrow \quad ext{complex when } 4\Delta > au^2.$$

Consider

$$\begin{cases} \dot{x} = x - y, \\ \dot{y} = x + y, \end{cases} \quad \mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} \tau = 2, \\ \Delta = 2, \end{aligned} \quad \Rightarrow \quad \begin{aligned} \lambda_1 = 1 + i, \\ \lambda_2 = 1 - i. \end{aligned}$$

The eigenvectors are

$$\begin{pmatrix} 0\\0 \end{pmatrix} = \begin{pmatrix} 1-\lambda & -1\\ 1 & 1-\lambda \end{pmatrix} \begin{pmatrix} u\\w \end{pmatrix} = \begin{pmatrix} (1-\lambda)u - w\\u + (1-\lambda)w \end{pmatrix} \Rightarrow w = (1-\lambda)u.$$
$$\lambda_1 = 1 - i \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 1\\-i \end{pmatrix}, \quad \lambda_2 = 1 - i \Rightarrow \mathbf{v}_2 = \begin{pmatrix} 1\\i \end{pmatrix}.$$

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5.2 Classification...: Example 2, time dependence The time dependence (with initial condition is given by

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2 = e^t \left[c_1 e^{it} \begin{pmatrix} 1 \\ -i \end{pmatrix} + c_2 e^{-it} \begin{pmatrix} 1 \\ i \end{pmatrix} \right],$$

and the initial condition, $\mathbf{x}_0 = (x_0, y_0)$, then gives $x_0 = c_1 + c_2$ and $y_0 = -ic_1 + ic_2 \implies c_{1,2} = (x_0 \pm iy_0)/2$. We find

$$\begin{aligned} \mathbf{x}(t) &= \frac{x_0 + iy_0}{2} e^t e^{it} \begin{pmatrix} 1 \\ -i \end{pmatrix} + \frac{x_0 - iy_0}{2} e^t e^{-it} \begin{pmatrix} 1 \\ i \end{pmatrix} \\ &= \frac{x_0}{2} e^t \begin{pmatrix} e^{it} + e^{-it} \\ ie^{-it} - ie^{it} \end{pmatrix} + \frac{iy_0}{2} e^t \begin{pmatrix} e^{it} - e^{-it} \\ -ie^{-it} - ie^{it} \end{pmatrix} \\ &= \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^t \cos t + \begin{pmatrix} -y_0 \\ x_0 \end{pmatrix} e^t \sin t. \end{aligned}$$

This is a spiral out— combination of exponential growth and oscillation.

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5.2 General classification

With complex eigenvalues, $\lambda_{1,2} = \alpha \pm i\omega$, there are three possibilities:

- $\alpha < 0$: spiral inwards,
- $\alpha = 0$: center (as for the harmonic oscillator)
- $\alpha > 0$: outward spiral.

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5.2 General classification



5.2 General classification



• If $\Delta < 0$ the eivenvalues are real with opposite signs: saddle point.

- if $\Delta > 0$: stable if $\tau < 0$ (node or spiral)
 - when $4\Delta < \tau^2$: eigenvalues real with the same sign (nodes)
 - when $4\Delta > \tau^2$: complex conjugate (spirals and centers)

Example: Classify the fixed point $\mathbf{x}^* = 0$ for $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$! Solution: the matrix has $\Delta = 1 \times 4 - 2 \times 3 = -2$; the fixed point is therefore a saddle point.

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5.2 Classification... For equal eigenvalues!

Equal eigenvalues and only a single eigenvector, \mathbf{v}_1 .

Try with
$$\mathbf{x}(t) = te^{\lambda t}\mathbf{v}_1 + e^{\lambda t}\mathbf{v}_2$$
 and plug it into $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$:
 $e^{\lambda t}\mathbf{v}_1 + \lambda te^{\lambda t}\mathbf{v}_1 + \lambda e^{\lambda t}\mathbf{v}_2 = te^{\lambda t}\mathbf{A}\mathbf{v}_1 + e^{\lambda t}\mathbf{A}\mathbf{v}_2$

After dividing by $e^{\lambda t}$ we get two equations:

$$\begin{aligned} \lambda \mathbf{v}_1 &= \mathbf{A} \mathbf{v}_1, \\ \mathbf{v}_1 + \lambda \mathbf{v}_2 &= \mathbf{A} \mathbf{v}_2. \end{aligned}$$

where \mathbf{v}_2 is given by the solution to

$$[\mathbf{A} - \lambda \mathbf{I}]\mathbf{v}_2 = \mathbf{v}_1.$$

This means that $\mathbf{x}_2(t) = te^{\lambda t}\mathbf{v}_1 + e^{\lambda t}\mathbf{v}_2$ will also be a solution and the general solution becomes

$$\mathbf{x}(t) = c_1 e^{\lambda t} \mathbf{v}_1 + c_2 (t e^{\lambda t} \mathbf{v}_1 + e^{\lambda t} \mathbf{v}_2), \quad \Rightarrow \quad \mathbf{x}_0 = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2.$$

It follows that $c_2 \neq 0$ only if \mathbf{x}_0 is not parallel to \mathbf{v}_1 . Solution:

$$\mathbf{x}(t) = e^{\lambda t} [\mathbf{x}_0 + c_2 t \mathbf{v}_1].$$