II Two-dimensional flows

Start with linear systems in Ch. 5
introduce nonlinearities in Ch. 6.

Two kinds of 2D problems:

- variables are different things, rabbits, foxes,
- cases where $n = 2$ comes from the second order equation, $\ddot{x} = \ldots$.

5 Linear systems

General twodimensional:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2), \\ \dot{x}_2 &= f_2(x_1, x_2). \end{aligned}$$

Specialize to linear systems (with $x_1 \to x$ and $x_2 \to y$):

$$\begin{aligned} \dot{x} &= ax + by, \\ \dot{y} &= cx + dy. \end{aligned}$$

Matrix notation:

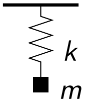
$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

Graphical analysis in 2D—(a) harmonic oscillator

Consider $m\ddot{x} = -kx$:

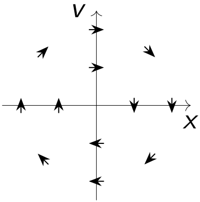
$$\begin{cases} \dot{x} = v, \\ \dot{v} = -\omega^2 x, \quad \omega^2 = k/m. \end{cases}$$

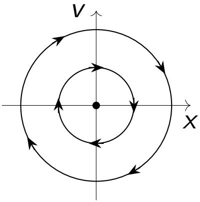


which implies

$$(\dot{x}, \dot{v}) = (v, -\omega^2 x), \quad \text{a vector to each point } (x, v).$$

To find a trajectory, put an imaginary particle at (x_0, v_0) and see how it is carried around by the flow.
"Phase portrait"





Graphical analysis in 2D—(b) uncoupled equations

Consider

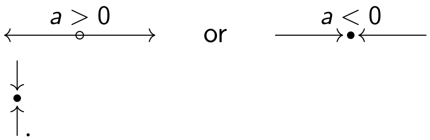
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

$$\begin{cases} \dot{x} = ax \\ \dot{y} = -y \end{cases}$$

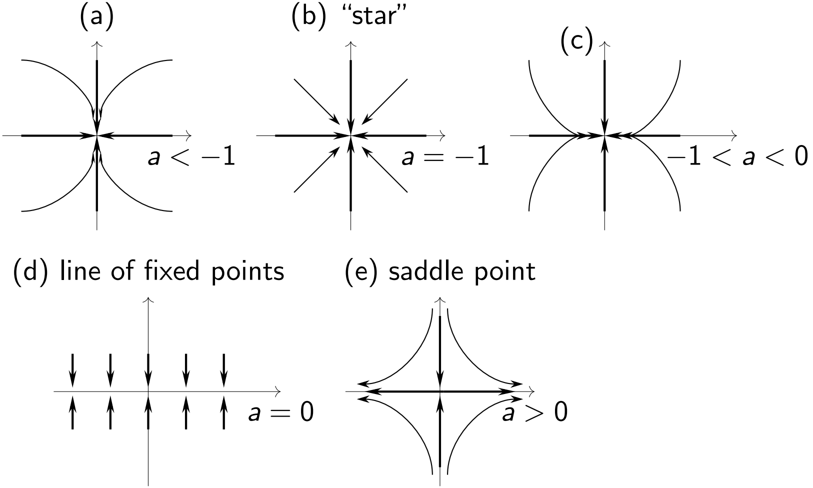
Solved separaterly:

$$x(t) = x_0 e^{at},$$

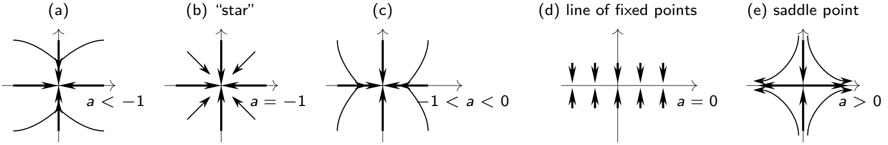
$$y(t) = y_0 e^{-t},$$



Uncoupled equations. . . many different possible behaviors



Uncoupled equations... many different possible behaviors



Stability language:

- (i) globally attracting as in a-c: all that starts near $\mathbf{x}^*$ approach it as $t \to \infty$.
- (ii) Liapunov stable — starting close to $\bar{x}^*$ remains close for all times.

More terms:

- neutrally stable: (ii) but not (i). Examples: (d) and (0,0) in the harmonic oscillator.
- stable: both (i) and (ii).
- unstable: neither (i) nor (ii).

5.2 Classification of linear systems

In the uncoupled case the $x-$ and $y-$ axes had straight trajectories. Starting on one of them we stayed on it forever.

Now with $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, look for trajectories $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$.
Put into $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$:

$$\lambda e^{\lambda t}\mathbf{v} = e^{\lambda t}\mathbf{A}\mathbf{v} \quad \Rightarrow \quad \lambda\mathbf{v} = \mathbf{A}\mathbf{v} \quad \text{— eigenvalue equation}$$

To determine the eigenvectors:

$$\left[\mathbf{A} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\mathbf{v} = 0,$$

which is solved through the charcteristic equation,

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

$$\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0 \quad \Rightarrow \quad (a-\lambda)(d-\lambda) - bc = 0.$$

$$\lambda^2 - \tau\lambda + \Delta = 0, \quad \text{where } \tau = \mathrm{Tr}\mathbf{A} = a + d, \quad \Delta = \det\mathbf{A} = ad - bc.$$

5.2 Classification of linear systems. . . cont'd

With $\tau = \mathrm{Tr}\mathbf{A} = a + d$ and $\Delta = \det\mathbf{A} = ad - bc$,

$$(a-\lambda)(d-\lambda) - bc = 0 \quad \Rightarrow \quad \lambda^2 - \lambda(a+d) + (ad-bc) = 0,$$

this may be rewritten

$$\lambda^2 - \tau\lambda + \Delta = 0,$$

with solutions

$$\lambda_1 = \frac{\tau}{2} + \frac{1}{2}\sqrt{\tau^2 - 4\Delta}, \quad \lambda_2 = \frac{\tau}{2} - \frac{1}{2}\sqrt{\tau^2 - 4\Delta}.$$

Two eigenvectors: $\mathbf{A}\mathbf{v}_1 = \lambda_1\mathbf{v}_1$, $\mathbf{A}\mathbf{v}_2 = \lambda_2\mathbf{v}_2$.

The eigenvalues are typically distinct, $\lambda_1 \neq \lambda_2$. The eigenvectors are then independent and span the plane. Any initial condition can then be written

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2, \quad \Rightarrow \quad \mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2.$$

5.2 Classification of linear systems...examples

Ex. 1 (5.2.1 and 5.2.2): Solve

$$\begin{cases} \dot{x} = x + y, \\ \dot{y} = 4x - 2y. \end{cases}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \Rightarrow \tau = -1, \quad \Delta = -6 \quad \text{which gives } \lambda_1 = 2, \quad \lambda_2 = -3.$$

Find eigenvectors, $\mathbf{v} = \begin{pmatrix} u \\ w \end{pmatrix}$:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} (1-\lambda)u + w \\ 4u - (2+\lambda)w \end{pmatrix}.$$

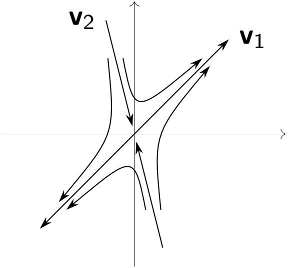
This gives $w = (\lambda - 1)u$ which can be used to determine the (unnormalized) eigenvectors:

$$\lambda_1 = 2 \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda_2 = -3 \Rightarrow \mathbf{v}_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

With $\lambda_1 = 2$ and $\lambda_2 = -3$ together with

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

we get the phase portrait



This is a saddle point!

The *stable manifold* is the set of points along $\pm\mathbf{v}_2$.

The *unstable manifold* is the set of points along $\pm\mathbf{v}_1$.

5.2 Classification of linear systems: Example 2

$$\lambda_{1,2} = \frac{\tau}{2} \pm \frac{1}{2}\sqrt{\tau^2 - 4\Delta}, \quad \Rightarrow \quad \text{complex when } 4\Delta > \tau^2.$$

Consider

$$\begin{cases} \dot{x} = x - y, \\ \dot{y} = x + y, \end{cases} \quad \mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \Rightarrow \quad \begin{array}{c} \tau = 2, \\ \Delta = 2, \end{array} \quad \Rightarrow \quad \begin{array}{c} \lambda_1 = 1 + i, \\ \lambda_2 = 1 - i. \end{array}$$

The eigenvectors are

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} (1-\lambda)u - w \\ u + (1-\lambda)w \end{pmatrix} \Rightarrow w = (1-\lambda)u.$$

$$\lambda_1 = 1 - i \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}, \quad \lambda_2 = 1 - i \Rightarrow \mathbf{v}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

5.2 Classification. . . : Example 2, time dependence

The time dependence (with initial condition is given by

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 = e^t \left[c_1 e^{it} \begin{pmatrix} 1 \\ -i \end{pmatrix} + c_2 e^{-it} \begin{pmatrix} 1 \\ i \end{pmatrix} \right],$$

and the initial condition $\mathbf{x}_0 = (x_0, y_0))$ then gives
$x_0 = c_1 + c_2$ and $y_0 = -ic_1 + ic_2 \quad \Rightarrow \quad c_{1,2} = (x_0 \pm y_0)/2.$

We find

$$\begin{aligned}
\mathbf{x}(t) &= \frac{x_0 + iy_0}{2} e^t e^{it} \begin{pmatrix} 1 \\ -i \end{pmatrix} + \frac{x_0 - iy_0}{2} e^t e^{-it} \begin{pmatrix} 1 \\ i \end{pmatrix} \\
&= \frac{x_0}{2} e^t \begin{pmatrix} e^{it} + e^{-it} \\ ie^{-it} - ie^{it} \end{pmatrix} + \frac{iy_0}{2} e^t \begin{pmatrix} e^{it} - e^{-it} \\ -ie^{-it} - ie^{it} \end{pmatrix} \\
&= \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^t \cos t + \begin{pmatrix} -y_0 \\ x_0 \end{pmatrix} e^t \sin t.
\end{aligned}$$

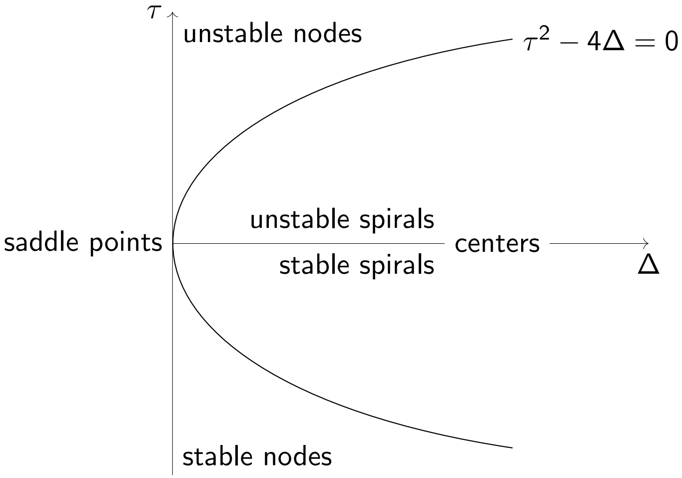
This is a spiral out— combination of exponential growth and oscillation.

5.2 General classification

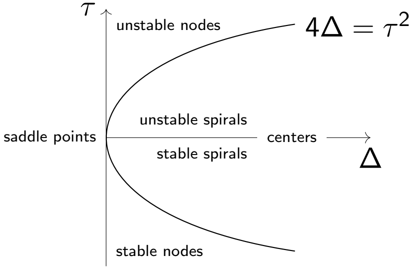
With complex eigenvalues, $\lambda_{1,2} = \alpha \pm i\omega$, there are three possibilities:

- $\alpha < 0$: spiral inwards,
- $\alpha = 0$: center (as for the harmonic oscillator)
- $\alpha > 0$: outward spiral.

5.2 General classification



5.2 General classification



- If $\Delta < 0$ the eivenvalues are real with opposite signs: saddle point.
- if $\Delta > 0$: stable if $\tau < 0$ (node or spiral)
  - when $4\Delta < \tau^2$: eigenvalues real with the same sign (nodes)
  - when $4\Delta > \tau^2$: complex conjugate (spirals and centers)

Example: Classify the fixed point $\mathbf{x}^* = 0$ for $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$!

Solution: the matrix has $\Delta = 1 \times 4 - 2 \times 3 = -2$; the fixed point is therefore a saddle point.

5.2 Classification. . . For equal eigenvalues!

Equal eigenvalues and only a single eigenvector.

Try with $\mathbf{x}(t) = te^{\lambda t}\mathbf{v}_1 + e^{\lambda t}\mathbf{v}_2$ and plug it into $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$:

$$e^{\lambda t}\mathbf{v}_1 + \lambda te^{\lambda t}\mathbf{v}_1 + \lambda e^{\lambda t}\mathbf{v}_2 = te^{\lambda t}\mathbf{A}\mathbf{v}_1 + e^{\lambda t}\mathbf{A}\mathbf{v}_2.$$

After dividing by $e^{\lambda t}$ we get two equations:

$$\begin{aligned} \lambda\mathbf{v}_1 &= \mathbf{A}\mathbf{v}_1, \\ \mathbf{v}_1 + \lambda\mathbf{v}_2 &= \mathbf{A}\mathbf{v}_2. \end{aligned}$$

whereas the second is given by the solution to

$$[\mathbf{A} - \lambda\mathbf{I}]\mathbf{v}_2 = \mathbf{v}_1.$$

This means that $\mathbf{x}_2(t) = te^{\lambda t}\mathbf{v}_1 + e^{\lambda t}\mathbf{v}_2$ will also be a solution and the general solution becomes

$$\mathbf{x}(t) = c_1 e^{\lambda t}\mathbf{v}_1 + c_2(te^{\lambda t}\mathbf{v}_1 + e^{\lambda t}\mathbf{v}_2), \quad \Rightarrow \quad \mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2.$$

It follows that $c_2 \neq 0$ only if $\mathbf{x}_0$ is not parallel to $\mathbf{v}_1$. Solution:

$$\mathbf{x}(t) = e^{\lambda t}[\mathbf{x}_0 + c_2 t\mathbf{v}_1].$$