2.4 Linear stability analysis

We now turn to a mathematical method to determine the character of a fixed point.

Notation for a fixed point: x^* with $f(x^*) = 0$.

Consider a small deviation from a fixed point. Does this deviation grow or decay with time?

$$\begin{aligned} \eta(t) &= x(t) - x^*, \quad \dot{x} = f(x), \\ \dot{\eta} &= \dot{x}, \quad \text{since } \dot{x}^* = 0. \end{aligned}$$

This gives. . . with a Taylor expansion

$$\dot{\eta} = f(x) = f(x^* + \eta) = f(x^*) + \eta f'(x^*) + O(\eta^2),$$

If $f'(x^*) \neq 0$:

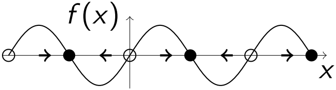
$$\dot{\eta} = \eta f'(x^*) \quad \text{— linearization about } x^*.$$

2.4 Linear stability analysis... cont'd

The perturbation η...

- grows exponentially if $f'(x^*) > 0$, unstable fixed point,
- decays if $f'(x^*) < 0$, stable fixed point.

Compare with the graphical analysis:



(If $f'(x^*) = 0$ we need a nonlinear analysis to determine the stability.)

From $\dot{\eta} = \eta f'(x^*)$ we get

$$\eta(t) = \eta_0 e^{f'(x^*)t}, \quad \text{characteristic time scale } 1/|f'(x^*)|.$$

This is the time required for $x(t)$ to vary significantly in the neighborhood of x^*.

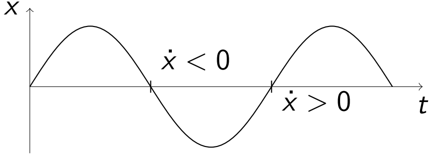
2.5 Existence and uniqueness

One can run into troubles if $f(x)$ behaves badly, e.g. $f'(0) \to \infty$.

This is seldom a problem in physics.

2.6 Impossibility of oscillations

An oscillation means that $\dot{x}$ can have different signs for the same x.



This is not possible if the dynamics is governed by

$$\dot{x} = f(x),$$

we thus conclude that there cannot be oscillations in a first order system.

3. Bifurcations—the splitting into two branches

One-dimensional problems: $x(t) \to$const or $x(t) \to \pm\infty$.
Boring!!

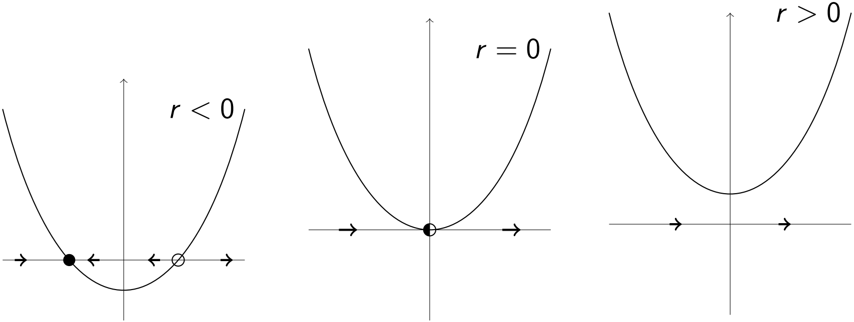
What is interesting about one-dimensional systems is the *dependence on parameters*!

3.1 Saddle node bifurcation

("saddle" makes sense in 2D, p. 242.)

The basic mechanism by which fixed points are created and destroyed.

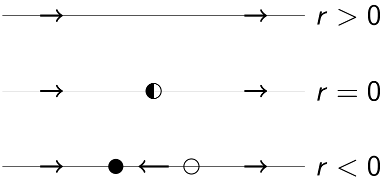
$$\dot{x} = r + x^2, \quad \text{parameter } r.$$

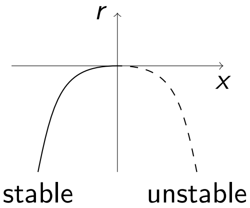


stable and unstable   half-stable   no fixed point

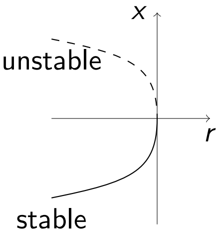
3.1 Saddle node bifurcation. . . cont'd

This can be illustrated in different ways:





More common to rotate the diagram:



3.1 Saddle node bifurcation. . . cont'd

Two more points:

- Note that $f(x) = r - x^2$ also has a saddle-point bifurcation.
- The same is true for any function with a minimum or maximum.

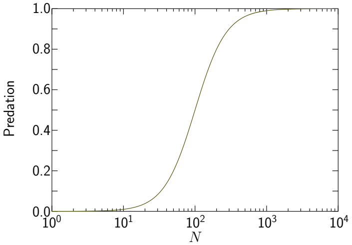
An example that illustrates this is the insect outbreak model that we are turning to next.

3.7 Insect outbreak

The dynamics is given by

$$\frac{dN}{dt} = r_B N \left(1 - \frac{N}{K_B}\right) - p(N),$$

where $p(N)$ represents predation by birds.



We now specialize to the following form for predation:

$$p(N) = \frac{BN^2}{A^2 + N^2}$$

which has a change from low to high predation at an approximate threshold value $N_c = A$. The dynamics now becomes

$$\frac{dN}{dt} = r_B N \left(1 - \frac{N}{K_B}\right) - \frac{BN^2}{A^2 + N^2}.$$

3.7 Insect outbreak. . . Nondimensionalisation

With four parameters, r_B, K_B, B, and A and it is difficult to analyze the model. To simplify things we introduce the dimensionless quantities

$$u = \frac{N}{A}, \quad r = \frac{Ar_B}{B}, \quad q = \frac{K_B}{A}, \quad \tau = \frac{Bt}{A},$$

which leads to the equation

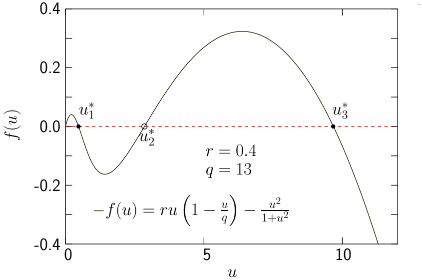
$$\frac{du}{d\tau} = ru\left(1 - \frac{u}{q}\right) - \frac{u^2}{1+u^2} = f(u; r, q).$$

- Two parameters r and q which are pure numbers.
- The time scale is also changed.
- (Also other possible ways to make things dimensionless.)

3.7 Insect outbreak. . . Fixed points

- The fixed points are where $f(u) = 0$.
- Three nontrivial solutions: u_1^*, u_2^*, u_3^*.
- Stable of unstable? Examine $f'(u^*)$!

  - Here $du/d\tau = f(u; r, q)$.
  - Consider the sign of $\partial f/\partial u$.
    - ★ u_1: $f'(u_1) < 0$ — stable.
    - ★ u_2: $f'(u_2) > 0$ — unstable.
    - ★ u_3: $f'(u_3) < 0$ — stable.



How will things change with the parameters, r and q?

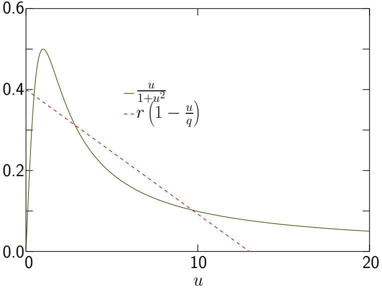
3.7 Insect outbreak... Innovative graphical solution

The steady states are solutions of

$$f(u; r, q) = 0 \Rightarrow ru\left(1 - \frac{u}{q}\right) = \frac{u^2}{1 + u^2}.$$

Look for the solutions to

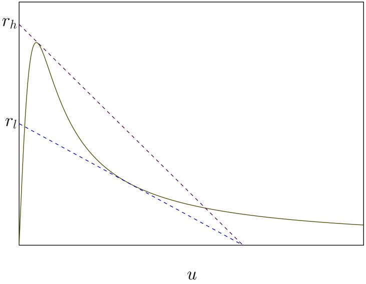
$$r\left(1 - \frac{u}{q}\right) = \frac{u}{1 + u^2}.$$



- The right hand side is a curve with a non-trivial shape, independent of r and q.
- The left hand side depends on the parameters but is simply a straight line from $(0, r)$ to $(q, 0)$.
- The solutions are given by the intersections of these curves. Different r and q give different straight lines and either one or three solutions.

3.7 Insect outbreak. . . The effect of changing the parameters

Three solutions for $r_l \le r \le r_h$. Otherwise only one solution.



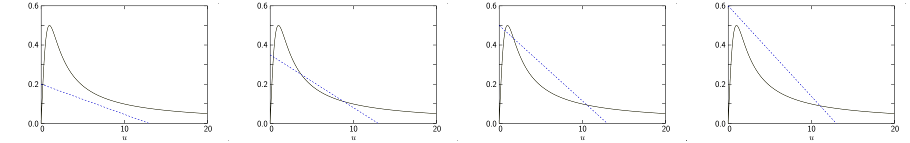
What is the effect of a gradual change of the parameter r?

It turns out that we get a hysteretical behavior.

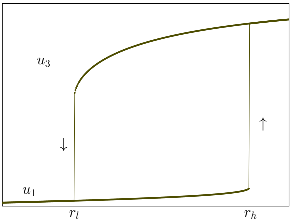
(It could be that r changes gradually because of changes in the environment.)

3.7 Insect outbreak. . . The effect of changing the parameters

Consider a gradual change of r from a small value to $r > r_h$ and back again!



- When $r > r_h$ the fixed point u_1^* disappears. The system jumps to u_3^*.
- When $r < r_h$ not much happens; the system remains at u_3^*.
- When $r < r_l$ the fixed point u_3^* disappears and the system jumps back to u_1^*.



3.7 Insect outbreak... Concern for the environment

The behavior above show the mechanism behind a tipping point:

- Things first just change gradually and slowly,
- When a parameter exceeds some critical value the system jumps to a different fixed point.
- Even if the parameter could be lowered below that critical value, the system could be stuck at this new fixed point.

Next lecture (Friday)

5. Linear systems with $n = 2$.
   - harmonic oscillator,
   - uncoupled equations,
   - classifications of linear systems,

Compare with the classification:

	$n = 1$	$n = 2$	$n \geq 3$	$n \gg 1$	continuum
lin-ear	growth, decay or equilibrium	oscillations		solid state physics	elasticity, wave eqs
non-lin-ear	Fixed points, bifurcations	pendulum, limit cycles	chaos, strange attractors	*research problems of today*	