Where to find these pages

The lecture notes from my teaching are at our local web server:

www.tp.umu.se/ModSim/L-notes

(where “tp” stands for “theoretical physics”.)

1. Introduction

Dynamical systems

Strogatz, Nonlinear dynamics and chaos, Ch. 1–3, 5–7.

- Fun and interesting.
- All kinds of applications! (Epidemiology, geophysics, fluid dynamics, materials science, engineering).

Strogatz Ch 1.0 and 1.1: Historical notes—read yourselves.

1.2 The importance of being nonlinear

Four important points!

a. Two kinds of dynamical systems:
   - Differential equations, $\dot{x} = f(x), \quad \to x(t)$.
   - Difference equations, $x_{n+1} = f(x_n), \quad \to x_0, x_1, x_2, \ldots$.
b. We discuss ordinary differential equations:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, \ldots, x_n), \\ \dot{x}_2 &= f_2(x_1, x_2, \ldots, x_n), \\ &\cdot \\ \dot{x}_n &= f_n(x_1, x_2, \ldots, x_n), \end{aligned}$$

   - Not partial differential equations $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$.
   - Not stochastic differential equation. (That comes in "Stochastic simulations", 22-29/9.)

- We discuss first order equations.
  Therefore: rewrite second order equations, $\ddot{x} = ax$:
  Let $x \to x_1$ and $x_2 = \dot{x}_1 \quad \Rightarrow \quad \dot{x}_2 = \ddot{x}_1$.
  Written in standard form:

$$\begin{cases} \dot{x}_2 = a x_1, \\ \dot{x}_1 = x_2. \end{cases}$$

  Two equations $\Rightarrow$ $n = 2$.

4. Linear or nonlinear? Linear:

$$\left.\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= a x_1 + b x_2, \end{aligned}\right\} \text{linear—all } x \text{ appear to first order only}$$

Nonlinear:

$$\ddot{x} = -\frac{g}{L} \sin x, \quad \text{nonlinear—since "}\sin x\text{" is a nonlinear function.}$$

This is the equation for the pendulum. From $F = m\dot{v}$ with

$$\left.\begin{aligned} v &= L\dot{\theta}, \\ F &= -mg \sin\theta, \end{aligned}\right\} \text{from the geometry of the problem.}$$

(d) Linear or nonlinear?... cont'd

$$\ddot{\theta} = -\frac{g}{L} \sin\theta.$$

How do we solve this?

- **Cheat:** Consider small θ only $\quad\Rightarrow\quad \sin\theta \to \theta \quad\Rightarrow\quad$ linear problem

$$\ddot{\theta} = -\frac{g}{L}\theta.$$

- **Here:** Keep the nonlinearity! Extract information about the system with graphical methods without actually solving the system.

Why are nonlinear problems so difficult?

Consider linear systems:

- Linear systems can be broken into parts...
- each part can be solved separately...
- and finally recombined to get the answer.

This gives a fantastic simplification of complex problems,
e.g. with Fourier analysis.

This isn't possible with nonlinear problems and that is why they are so difficult.

1.3 Classify problems—n and linear/nonlinear

	$n = 1$	$n = 2$	$n \geq 3$	$n \gg 1$	continuum
linear	growth, decay or equilibrium	oscillations		solid state physics	elasticity, wave eqs
nonlinear	Fixed points, bifurcations	pendulum, limit cycles	chaos, strange attractors	*research problems of today*	

- The simplest systems are in the upper left corner.
- Also familiar is the upper right corner. Partial differential equations, Maxwell's equations, Schrödinger's equation. (Infinite continuum.)
- This course focuses on the lower left corner. Start with $n = 1$ and move to the right.
- The lower right corner—many problems for today's research.

Part 1—One-dimensional flows, $n = 1$

The general case:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, \ldots, x_n), \\ &\vdots \\ \dot{x}_n &= f_n(x_1, x_2, \ldots, x_n). \end{aligned}$$

Now focus on only one variable:

$$\dot{x} = f(x).$$

Here

- $x(t)$ is a real-valued function of time.
- $f(x)$ is a smooth real-valued function of x.

Note:

- $f(x)$ may not depend on time,
- in our terminology $f(x, t)$ would be a two-dimensional system.

2.1 A geometric way of thinking

Pictures are often more helpful than formulas!

Consider

$$\dot{x} = \sin x, \quad x(0) \equiv x_0,$$

which can be solved analytically.

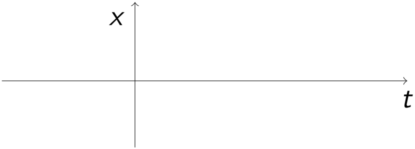
$$t = \ln \left| \frac{\csc x_0 + \cot x_0}{\csc x + \cot x} \right| .$$

That doesn't help much!

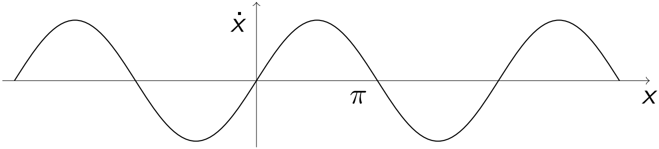
1. Suppose $x_0 = \pi/4$. Describe $x(t)$ qualitatively. What happens as $t \to \infty$?
2. For arbitrary x_0 what happens as $t \to \infty$?

Graphical analysis

We are most used at plotting $x(t)$:

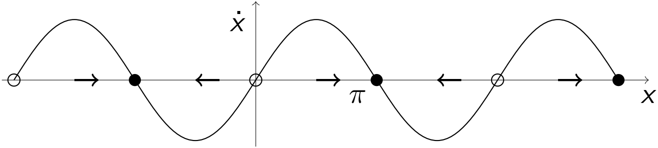


We will here use a different kind of plot: $\dot{x}$ vs x.



Here $\dot{x}$ represents the vector field on the line; the velocity at each given x

Put arrows on the axis to indicate the flow direction!

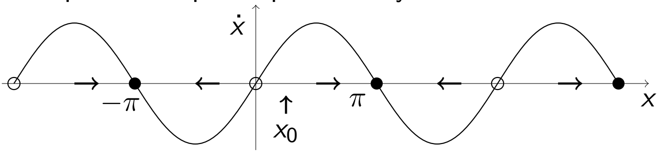


Points with $\dot{x} = 0$ — *fixed points*.

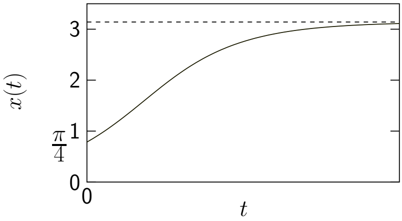
Two kinds of fixed points:

- Solid dots: *stable* fixed points—flow towards the fixed point.
- Open dots: *unstable* fixed points—flow out from the fixed point.

This picture helps to qualitatively understand the solution to $\dot{x} = \sin x$.



- A particle starting at $x = x_0 \equiv \pi/4$ moves to the right faster and faster until it reaches $x = \pi/2$.
- When $x > \pi/2$ it slows down and gradually approaches $x = \pi$.



The same method may be used for any starting point:

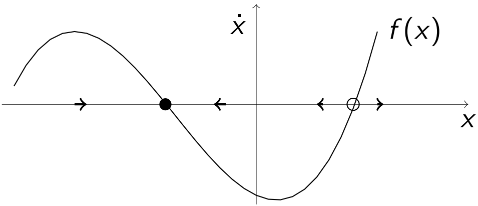
- With $x_0 = -0.5$ we would end up at $x = -\pi$,
- $x_0 = 3\pi/2$ takes us to $x = \pi$.

(Limitation:Not much *quantitative* information.)

2.2 Fixed points and stability

This idea can be used for any $\dot{x} = f(x)$:

- Draw the graph $f(x)$,
- construct the vector field,
- locate the fixed points.



A figure with the flows and the fixed points is called "phase portrait".

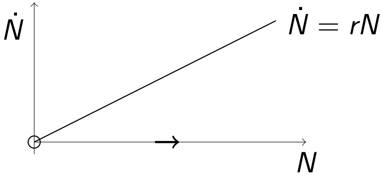
(This becomes even more useful and interesting in the two-dimensional case.)

2.3 Population growth

Models for population growth — births and deaths in proportion to the size of the population.

Simplest model for population growth (linear):

$$\dot{N} = rN, \quad N(t) \text{ — population at time } t.$$

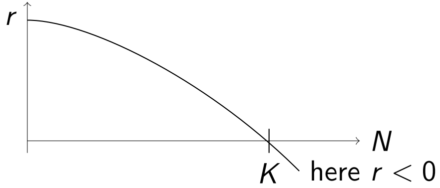


Solution: $N(t) = N_0 e^{rt}$. (Check: $dN/dt = N_0 r e^{rt} = rN(t)$).

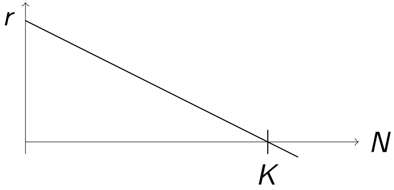
Not realistic for large times!

2.3 Population growth... cont'd

Every system has a *carrying capacity*, K.



Let the growth rate depend on N.



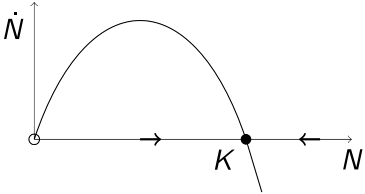
Mathematically convenient, the logistic equation:

$$\dot{N} = rN\left(1 - \frac{N}{K}\right).$$

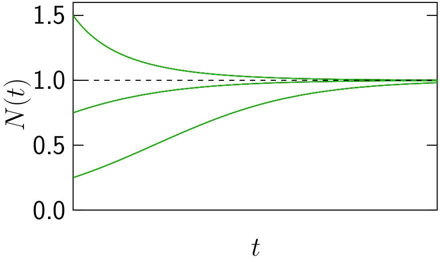
2.3 Population growth. . . cont'd

Graphical analysis of

$$\dot{N} = rN\left(1 - \frac{N}{K}\right).$$



Solutions for three different N_0:



Next lecture:

- 2.4 Linear stability analysis.
- (2.5 Existence and uniqueness.)
- 2.6 Impossibility of oscillations.
- 3.1 Bifurcations—how things can change with a change in a parameter.
- 3.7 Application: insect outbreak.